On the simulation of expanding spacetimes in trapped Bose-Einstein condensates

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Motivation

Low-energy phase fluctuations in fluid obey same evolution equations as massless scalar field in a certain curved spacetime

Universal features of quantum effects (e.g., adiabatic theorem)

Bose-Einstein condensates

\[ i\partial_t \hat{\Psi} = \left[ -\frac{\nabla^2}{2} + V_{\text{ext}} + g \hat{\Psi}^\dagger \hat{\Psi} \right] \hat{\Psi} \]
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Change interaction strength \( g(t) \)

Vary external trap potential \( V_{\text{ext}}(r, t) \)
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\]

Change interaction strength \( g(t) \)

Vary external trap potential \( V_{\text{ext}}(r, t) \)

Simple experiment: Tune \( g(t) \) over huge range

Measure density-density correlations
Effective Spacetime

Split field operator $\hat{\Psi} = \Psi_0 + \delta \hat{\Psi}$ into modulus and phase

$\Psi_0 = e^{i \Phi_0} \sqrt{\rho_0}$, \hspace{1em} \delta \hat{\psi} = \Psi_0 \left( \frac{\delta \hat{\rho}}{2 \rho_0} + i \delta \hat{\Phi} \right)$
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$$\Psi_0 = e^{i\Phi_0} \sqrt{\rho_0} , \quad \delta \hat{\psi} = \Psi_0 \left( \frac{\delta \hat{\varphi}}{2\rho_0} + i\delta \hat{\Phi} \right)$$

Bernoulli and continuity equations for background $\nu_0 = \nabla \Phi_0$

Equation of state $\nabla p = \rho_0 \nabla [V_{\text{ext}} - (\nabla^2 \sqrt{\rho_0})/2\sqrt{\rho_0} + g\rho_0]$
Effective Spacetime

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If $[\nabla^2 \sqrt{\rho_0}] / \sqrt{\rho_0}$ negligible: effective space-time

$$\partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \delta \hat{\Phi} = 0, \quad g_{\mu\nu} = \frac{1}{A_D c_s^2} \begin{pmatrix} c_s^2 - \nu_0^2 & \nu_0^i \\ \nu_0^j & -\delta_{ij} \end{pmatrix}$$

with sound velocity $c_s^2 = g\rho_0$, and prefactor $A_D = (c_s/g)^{2/(D-1)}$
Effective Spacetime

Split field operator \( \hat{\Psi} = \Psi_0 + \delta \hat{\Psi} \) into modulus and phase

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\]

Simulate aspects of cosmic quantum effects

Apply concepts from general relativity
Effective Spacetime

Split field operator $\hat{\Psi} = \Psi_0 + \delta \hat{\Psi}$ into modulus and phase

$$\Psi_0 = e^{i\Phi_0} \sqrt{\rho_0}, \quad \delta \hat{\psi} = \Psi_0 \left( \frac{\delta \hat{\varphi}}{2 \rho_0} + i \delta \hat{\Phi} \right)$$

Bernoulli and continuity equations for background $\mathbf{v}_0 = \nabla \Phi_0$

Equation of state $\nabla p = \rho_0 \nabla \left[ V_{\text{ext}} - \left( \nabla^2 \sqrt{\rho_0} \right) / 2 \sqrt{\rho_0} + g \rho_0 \right]$

If $\left[ \nabla^2 \sqrt{\rho_0} \right] / \sqrt{\rho_0}$ negligible: effective space-time

$$\partial_{\mu} \sqrt{-g} g^{\mu \nu} \partial_{\nu} \delta \hat{\Phi} = 0, \quad g_{\mu \nu} = \frac{1}{A D C_s^2} \left( \begin{array}{cc} c_s^2 - \mathbf{v}_0^2 & \mathbf{v}_0^i \\ \mathbf{v}_0^j & -\delta_{ij} \end{array} \right)$$

In experiments usually external confinement

i.e., discrete excitation spectrum

Background motion?
Background motion

Field equation for background with harmonic trap

\[ i\partial_t \Psi_0 = \left[ -\frac{\nabla^2}{2} + \frac{\omega^2(t)r^2}{2} + g(t)|\Psi_0|^2 \right] \Psi_0 \]

Transformation to (almost) comoving coordinates

\[ \mathbf{x} = \frac{\mathbf{r}}{b(t)} \text{ and } \psi_0 = e^{i\Phi} \Psi_0 b^{D/2} \]

\[ ib^2(t)\partial_t \psi_0 = \left[ -\frac{\nabla^2_x}{2} + f^2(t) \left( \frac{\mathbf{x}^2}{2} + g_0|\psi_0|^2 \right) \right] \psi_0 \]

with \[ f^2(t) = b^3 \frac{\partial^2}{\partial t^2} b + b^4 \omega^2(t) = \frac{g(t)}{g_0} b^{2-D} \]

Most of background motion contained in \[ f^2(t) \]
small density ripples near surface of condensate
Field equation for background with harmonic trap

Transformation to (almost) comoving coordinates

\[ \mathbf{x} = \mathbf{r} / b(t) \] and

\[ \psi_0 = e^{i\Phi} \Psi_0 / b^{D/2} \]

\[ ib^2(t) \partial_t \psi_0 = \left[ -\frac{\nabla^2}{2} x + f^2(t) \left( \frac{x^2}{2} + g_0 \left| \psi_0 \right|^2 \right) \right] \psi_0 \]

with

\[ f^2(t) = b^3 \frac{\partial^2}{\partial t^2} b + b^4 \omega^2(t) = \frac{g(t)}{g_0} b^{2-D} \]

Most of background motion contained in \( f^2(t) \)

small density ripples near surface of condensate

Note \[ \left[ \nabla^2_x \sqrt{\rho_0} \right] / \sqrt{\rho_0} \] needs to be small for effective spacetime
Quantum fluctuations

Self-adjoint operators

\[ \hat{\chi}_+ = e^{-i\Phi_0} \delta \hat{\psi} + e^{i\Phi_0} \delta \hat{\psi}^\dagger = \frac{\delta \hat{\varrho}}{\sqrt{\varrho_0}} \]

\[ \hat{\chi}_- = \frac{1}{2i} \left( e^{-i\Phi_0} \delta \hat{\psi} - e^{i\Phi_0} \delta \hat{\psi} \right) = \sqrt{\varrho_0} \delta \hat{\phi} \]
Quantum fluctuations

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Initial eigenmode expansion

\[ \hat{\chi}_\pm = h_{n\pm}(\vec{\chi}; t_0) \hat{X}_{n\pm}(t) \]

with duality

\[ \int h_n^+(\vec{x}) h_m^-(\vec{x}) = \delta_{nm} \]

evolution equations

\[ b^2 \frac{\partial}{\partial t} \hat{X}_n^- = -\frac{1}{2} A_{nm}(t) \hat{X}_m^+ - V_{nm}(t) \hat{X}_m^- \]

\[ b^2 \frac{\partial}{\partial t} \hat{X}_n^+ = 2B_{nm}(t) \hat{X}_m^+ + V_{nm}(t) \hat{X}_m^- \]
Quantum fluctuations

Self-adjoint operators

\[
\hat{\chi}_+ = e^{-i\Phi_0} \delta \hat{\psi} + e^{i\Phi_0} \delta \hat{\psi}^\dagger = \frac{\delta \hat{\rho}}{\sqrt{\mathcal{Q}_0}}
\]

\[
\hat{\chi}_- = \frac{1}{2i} \left( e^{-i\Phi_0} \delta \hat{\psi} - e^{i\Phi_0} \delta \hat{\psi} \right) = \sqrt{\mathcal{Q}_0} \delta \hat{\phi}
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with duality

\[
\int h_{n}^+(\mathbf{x}) h_{m}^-(\mathbf{x}) = \delta_{nm}
\]

evolution equations are diagonal at \( t_0 \) (when \( \mathcal{V}_{nm} = 0 \))

\[
b^2 \frac{\partial}{\partial t} \hat{X}_n^- = -\frac{1}{2} \mathcal{A}_{nm}(t) \hat{X}_m^+ - \mathcal{V}_{nm}(t) \hat{X}_m^- \overset{t=t_0}{=} -\frac{1}{2} \mathcal{A}_n \hat{X}_n^+
\]

\[
b^2 \frac{\partial}{\partial t} \hat{X}_n^+ = 2\mathcal{B}_{nm}(t) \hat{X}_m^- + \mathcal{V}_{nm}(t) \hat{X}_m^+ \overset{t=t_0}{=} 2\mathcal{B}_n \hat{X}_n^-
\]
Coupling matrices

\[ A_{nm} = \int h_n^+ \left( -\frac{\nabla^2}{2} + 2f^2g_0\varrho_0 + \frac{\nabla^2 x\sqrt{\varrho_0}}{2\sqrt{\varrho_0}} \right) h_m^+ \]

\[ B_{nm} = \int h_n^- \left( -\frac{\nabla^2}{2} + \frac{\nabla^2 x\sqrt{\varrho_0}}{2\sqrt{\varrho_0}} \right) h_m^- \]

\[ V_{nm} = \int h_n^+ \left[ \nu_0 \nabla x + \frac{\nabla x\nu_0}{2} \right] h_m^- \]

Decoupling of modes \( n \) and \( m \) if
Coupling matrices

\[ A_{nm} = \int h_n^+ \left( -\frac{\nabla_x^2}{2} + 2 f^2 g_0 \varrho_0 + \frac{\nabla_x^2 \sqrt{\varrho_0}}{2 \sqrt{\varrho_0}} \right) h_m^+ \]

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Decoupling of modes \( n \) and \( m \) if

First integral dominated by \( f^2 g_0 \varrho_0 \)
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First integral dominated by \( f^2 g_0 \varrho_0 \)

Time-dependence of \( \varrho_0 \) must separate in support of modes
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Decoupling of modes \( n \) and \( m \) if

First integral dominated by \( f^2 g_0 \varrho_0 \)

Time-dependence of \( \varrho_0 \) must separate in support of modes

\[ \Rightarrow \text{Centre of the trap and low energies} \]

Background motion described by \( f^2(t) \)
Thomas-Fermi approximation

Evolution equation for low modes ($\hat{\chi}_- = \sqrt{\rho_0} \delta \hat{\phi}$, $d\tau = dt/b^2$)

\[
\left[ \frac{\partial^2}{\partial \tau^2} - 2 \frac{\partial \ln f}{\partial \tau} \frac{\partial}{\partial \tau} + f^2 A_n B_n \right] \delta \hat{\phi}_n = 0
\]

Same as for a (massless) scalar field mode in a (flat) Friedmann-Robertson-Walker-Lemaître spacetime
Thomas-Fermi approximation

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Apply horizon concept from cosmology etc.
Thomas-Fermi approximation

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Problem: time \( \tau \) and not laboratory time \( t \)
Faster than exponential sweep of \( g(t) \) required in static trap
Thomas-Fermi approximation

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Faster than exponential sweep of \(g(t)\) required in static trap

(Static) Thomas-Fermi profile

\[
\mu_0 = \frac{\omega^2 r^2}{2} + g \varrho_0 , \quad c_s^2 = g \varrho_0
\]

Sound velocity

Comparison \(r_{TF}^2 = \frac{2}{\omega^2} c_s^2 (r = 0)\)
Exponential sweep, $g = \omega = e^{-2\gamma t}$, $\gamma = 0.1$
Summary and conclusions

Universal features of quantum effects

Adiabatic theorem

Quantum fluctuations in trapped Bose-Einstein condensate

Effective spacetime for low-lying excitations $\Omega_n \ll \mu_0$

Intermode coupling for $\Omega_n \gtrsim \mathcal{O}(\mu_0)$

For realistic variations $g(t)$:

Need to change trap frequency $\omega(t)$ as well or adiabaticity not violated in effective space-time regime