Hawking Radiation from Black Holes of Constant Negative Curvature via Gravitational Anomalies

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Hawking radiation from the cancellation of gauge and gravitational anomalies near the event horizon


Simplification for asymptotically flat spacetimes


Motivations

1. Black holes of non-spherical topology
2. Black hole of non-spherical topology conformally coupled to a scalar field
3. Asymptotically non-flat spacetimes
Topological Black Holes (TBHs)

- **Action**

\[ I = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left[ R + \frac{6}{l^2} \right], \]

in asymptotically AdS\(_4\).

- **Black hole**

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\sigma^2, \]

\[ f(r) = \frac{r^2}{l^2} - 1 - \frac{2\mu}{r}. \]

\(\mu > -\frac{l}{3\sqrt{3}}\) is a constant related to the mass of the black hole as

\[ M = \left(\mu + l/3\sqrt{3}\right)(\tilde{g} - 1), \]

and

\[ d\sigma^2 = d\theta^2 + \sinh^2 \theta d\varphi^2, \]

with \(\theta \geq 0\) and \(0 \leq \varphi < 2\pi\), is the line element of the two-dimensional manifold \(\Sigma\)

\[ \Sigma = H^2/\Gamma. \]

\(\Sigma\) is a compact surface of constant negative curvature and of genus \(\tilde{g} \geq 2\).
The simplest manifold $\Sigma$ is a compact surface of genus two

Fig. from [R. B. Mann, arXiv:gr-qc/9709039]

The horizons

$$f(r) = \frac{r^2}{l^2} - 1 - \frac{2\mu}{r} = 0.$$  

- $-\frac{1}{3\sqrt{3}} < \mu < 0$: an inner horizon $r_-$ and an outer horizon $r_+$.  
- $\mu \geq 0$: one horizon $r_h$. 

The horizons have the non-trivial topology of the manifold $\Sigma$. 

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Wave Equation of a Scalar Field in the Background of a TBH

Wave equation of a massive scalar field $\Phi$

\[
\left[ -\frac{1}{f} \frac{\partial^2}{\partial t^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 f \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sinh \theta} \frac{\partial}{\partial \theta} \left( \sinh \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sinh^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] \Phi = m^2 \Phi.
\]

- Separation of variables
- Wave modes

\[
\Phi_{\xi m} = \frac{R_\xi(t,r)}{r} P_{-\frac{1}{2} \pm i \xi}(\cosh \theta) e^{im\phi}.
\]

where

\[
m = 0, \pm 1, \pm 2, \pm 3, \ldots
\]

[N. L. Balazs and A. Voros, Phys. Rept. 143, 109 (1986)]

- $\Sigma = H^2/\Gamma$ is a compact surface of genus $\tilde{g} = 2$. Discrete spectrum. $\xi \geq 0$ takes discrete real values.

Generally for genus $\tilde{g} > 2$, there are no analytical results for the angular eigenvalues and for the angular eigenfunctions.
Consider matter given by a complex scalar field $\phi(x)$ in the background of a TBH of genus $\tilde{g} = 2$. Action

$$S = S_{\text{free}} + S_{\text{int}}.$$  

- $S_{\text{int}}$ includes a mass term, potential terms and (self-)interaction terms.
- $S_{\text{free}}$ is the free part of the action

$$S_{\text{free}} = -\frac{1}{2} \int d^4x \sqrt{-g} \phi^* \nabla^2 \phi.$$  

Partial wave decomposition

$$\phi(x) = \sum_{m=-\infty}^{+\infty} \frac{R_{\xi m}(t, r)}{r} \mathcal{Y}_\xi^m(\theta, \varphi),$$

where

$$\mathcal{Y}_\xi^m(\theta, \varphi) \equiv \left( \frac{2\pi}{\xi \tanh(\pi \xi)} \right)^{1/2} \frac{\Gamma(i\xi + \frac{1}{2})}{\Gamma(i\xi + m + \frac{1}{2})} P_{\frac{1}{2} + i\xi}(\cosh \theta) e^{im\varphi}.$$
Substitute the partial wave decomposition in $S_{free}$, $S_{int}$ and use the properties of the $\mathcal{Y}_m^\xi$.

Transform to the tortoise coordinate

$$\frac{dr_*}{dr} = \frac{1}{f(r)}.$$  

Consider a region near the outer event horizon $r_H$. In this region

$$r(r_*) \approx Ae^{2\kappa r_*} + r_H,$$

$$f(r(r_*)) \approx 2\kappa Ae^{2\kappa r_*}.$$  

In the original coordinates

$$S = \sum_{m=-\infty}^{\infty} -\frac{1}{2} \int dt dr R^*_{\xi m} \left[ -\frac{1}{f} \partial_t^2 + \partial_r (f \partial_r) \right] R_{\xi m}.$$  

Physics in a region near the event horizon can be described by an infinite collection of (1+1)-dimensional free massless complex scalar fields, each propagating in a (1+1)-dimensional spacetime

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2.$$
Hawking Radiation

- Ignore the ingoing (left-moving) modes in the region near the event horizon.
- A gravitational anomaly appears
  - [R. A. Bertlmann and E. Kohlprath, Annals Phys. 288, 137 (2001)]
  - [S. A. Fulling, Gen. Rel. Grav. 18, 609 (1986)]

\[ \nabla_\mu \tilde{T}^{\mu\nu} = \frac{\epsilon^{\nu\mu}}{96\pi \sqrt{-g^{(2)}}} \partial_\mu R^{(2)}. \]

The \( \nu = t \) component is written as

\[ \partial_r \left( \tilde{T}_t^r - \tilde{N}_t^r \right) = 0, \]

where

\[ \tilde{N}_t^r = \frac{1}{96\pi} \left( ff'' - \frac{f'^2}{2} \right). \]

- Solution

\[ \tilde{T}_t^r (r) = a_H + \tilde{N}_t^r (r) - \tilde{N}_t^r (r_H). \]


\[ \tilde{T}_t^r (r_H) = 0. \]

- Therefore

\[ \tilde{T}_t^r (r) = \tilde{N}_t^r (r) - \tilde{N}_t^r (r_H). \]
In the asymptotic limit $r \to \infty$

\[
\frac{\epsilon^{\nu \mu}}{96\pi \sqrt{-g(2)}} \partial_\mu R_{(2)} \to 0 ,
\]

\[
\partial_r \tilde{N}_t \to 0 ,
\]

\[
\tilde{N}_t \to -\frac{l^{-2}}{48\pi} .
\]

The energy flux at infinity is

\[
F = \tilde{T}_t(r \to \infty) - \tilde{N}_t(r \to \infty) = -\tilde{N}_t(r_H) ,
\]

which equals to

\[
F = \frac{1}{192\pi} f^{r2}(r_H) = \frac{\pi}{12} \left( \frac{\kappa}{2\pi} \right)^2 .
\]

A beam of massless blackbody radiation of temperature $T$ moving in the positive $r$ direction has a flux $\Phi = \frac{\pi}{12} T^2$.

The flux is equivalent to blackbody radiation with a temperature $T = \kappa/2\pi$.

Identified with the Hawking temperature of the (3+1)-dimensional TBH of genus $\tilde{g} = 2$ [L. Vanzo, Phys. Rev. D 56, 6475 (1997)].
Action \cite{Martinez2004}

\[ I[g_{\mu\nu}, \Psi] = \int d^4x \sqrt{-g} \left[ \frac{R + 6l^{-2}}{16\pi G} - \frac{1}{2} g^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi - \frac{1}{12} R \Psi^2 - \frac{2\pi G}{3l^2} \Psi^4 \right]. \]

Black hole with scalar “hair”

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 \left( d\theta^2 + \sinh^2 \theta d\phi^2 \right), \]

\[ f(r) = \frac{r^2}{l^2} - \left( 1 + \frac{G\mu}{r} \right)^2, \]

\[ \Psi(r) = \sqrt{\frac{3}{4\pi G}} \frac{G\mu}{r + G\mu}, \quad \mu > -l/4G. \]
Horizons

- $\mu \geq 0$

$$r_+ = \frac{l}{2} \left( 1 + \sqrt{1 + \frac{4G\mu}{l}} \right).$$

- $-\frac{l}{4} < G\mu < 0$

$$r_- = \frac{l}{2} \left( -1 + \sqrt{1 - \frac{4G\mu}{l}} \right),$$

$$r_-- = \frac{l}{2} \left( 1 - \sqrt{1 + \frac{4G\mu}{l}} \right),$$

$$r_+ = \frac{l}{2} \left( 1 + \sqrt{1 + \frac{4G\mu}{l}} \right).$$

$r_--$ and $r_+$ are event horizons, while $0 < r_-- < -G\mu < r_- < l/2 < r_+$. 
Consider a scalar field $\phi(x)$ in the background of the MTZ black hole of genus $\tilde{g} = 2$

$$S = -\frac{1}{2} \int d^4 x \sqrt{-g} \phi^* \nabla^2 \phi + S_{int},$$

where there is no interaction of $\phi(x)$ with $\Psi(r)$.

Perform a partial wave decomposition of $\phi$ in terms of the $Y^m_\xi$.

The dimensional reduction procedure proceeds as previously.
Hawking Radiation from the MTZ Black Hole

- Ignore the ingoing modes in the region near the event horizon.
- The $\nu = t$ component of the gravitational anomaly

$$\partial_r \left( \tilde{T}_t^r - \tilde{N}_t^r \right) = 0,$$

- Boundary condition
- In the asymptotic limit $r \to \infty$

$$\frac{e^{\nu \mu}}{96\pi \sqrt{-g_{(2)}}} \partial_\mu R_{(2)} \to 0, \quad \partial_r \tilde{N}_t^r \to 0, \quad \tilde{N}_t^r (r) \to -\frac{l^{-2}}{48\pi}.$$

- The energy flux at infinity is

$$F = \tilde{T}_t^r (r \to \infty) - \tilde{N}_t^r (r \to \infty) = -\tilde{N}_t^r (r_H),$$

$$F = \frac{1}{192\pi} f^r r^2 (r_H) = \frac{\pi}{12} \left( \frac{\kappa}{2\pi} \right)^2.$$

This form is equivalent to blackbody radiation with temperature $T = \kappa / 2\pi$. Identified with the Hawking temperature of the MTZ black hole.

Conclusions

- Modification of the covariant anomaly method for asymptotically non-flat spacetimes & application to a TBH of genus \( \tilde{g} = 2 \).
- Application to a TBH of genus \( \tilde{g} = 2 \) conformally coupled to a scalar field.
- The Hawking flux is universally determined only by the value of the gravitational anomaly on the event horizon.
- The backscattering of the Hawking radiation is ignored and its thermal spectrum is not proved.