Wormhole dynamics

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1. Introduction

Traversable wormholes were originally studied in static, spherically symmetric space-times [Morris & Thorne 1988].

What about dynamic cases? Simply inserting ad hoc time-dependent terms into the Morris-Thorne metric does not necessarily produce a traversable wormhole, because the metric become singular precisely at the throat.

A more geometrical generalization exists in terms of trapping horizons [Hayward 1999].

This allows a large body of theory developed for black holes to be applied to wormholes.

A Morris-Thorne wormhole throat, at a given static time, is a minimal surface in the static hypersurface, i.e. locally minimizing area among surfaces in the hypersurface. Their "flaring-out" condition expresses strict minimality [Visser 1995].

A natural generalization is a spatial surface which is minimal in some spatial hypersurface. Except in doubly marginal cases, this is a (future or past) trapped surface [Penrose 1965].

Since this is a generic condition, a generic wormhole must consist of a space-time region.

The boundaries of this region would be expected to be trapping horizons, i.e. composed of marginal surfaces, which are extremal in null hypersurfaces.

For a two-way traversable wormhole, there should be two temporal boundaries in mutual causal contact.

Prosaically, the wormhole consists of a tunnel between two mouths. In static cases, the tunnel shrinks away and the two mouths coincide as the throat.

Consequences: a Morris-Thorne wormhole throat is a double trapping horizon, which will generally bifurcate under a dynamic perturbation, such as someone crossing it.

This raises the issues of stability and, if stable, maintenance, i.e. returning a perturbed wormhole to a static state.

Also, since black holes may also be defined locally in terms of trapping horizons, it is possible for a wormhole to collapse to a black hole, or for a black hole to be converted to a traversable wormhole.

Henceforth assume spherical symmetry for simplicity and Einstein gravity for definiteness.

2. Geometry

Area A, area radius $r = \sqrt{A/4\pi}$, $A = 4\pi r^2$.

$$\text{A sphere is } \left\{ \begin{array}{c} \text{untrapped} \\ \text{marginal} \\ \text{trapped} \end{array} \right\} \text{ if } g^{-1}(dr) \text{ is } \left\{ \begin{array}{c} \text{spatial} \\ \text{null} \\ \text{temporal} \end{array} \right\}$$

$$\text{and } \left\{ \begin{array}{c} \text{future} \\ \text{past} \end{array} \right\} \text{trapped or marginal if } g^{-1}(dr) \text{ is } \left\{ \begin{array}{c} \text{future} \\ \text{past} \end{array} \right\} \text{causal.}$$

A hypersurface foliated by marginal spheres is a trapping horizon [Hayward 1994].

Preferred time vector $k = g^{-1}(*dr)$ [Kodama 1980]

where $*$ is the Hodge operator in the space normal to the spheres of symmetry:

$k \cdot dr = 0$, $g(k, k) = -g^{-1}(dr, dr)$.

The Kodama vector coincides with the static Killing vector of standard black holes such as Schwarzschild and Reissner-Nordström.

$$\text{Note that } k \text{ is } \left\{ \begin{array}{c} \text{temporal} \\ \text{null} \\ \text{spatial} \end{array} \right\} \text{ on } \left\{ \begin{array}{c} \text{untrapped} \\ \text{marginal} \\ \text{trapped} \end{array} \right\} \text{ spheres.}$$

Both k and the corresponding energy-momentum density $j = -g^{-1}(T \cdot k)$, where T denotes the energy-momentum-stress tensor, are conserved [Hayward 1994]:

$\nabla \cdot k = 0$, $\quad \nabla \cdot j = 0$, where ∇ denotes the covariant derivative operator and the second property uses the Einstein equation.

These Noether currents therefore admit Noether charges

$V = -\int_\Sigma \hat{*} \cdot k$, $\quad m = -\int_\Sigma \hat{*} \cdot j$, where $\hat{*}$ denotes the volume form times unit normal of a spatial hypersurface with regular centre.

The charges are found to be area volume and active gravitational mass [Misner & Sharp 1964]:

$V = \frac{4}{3}\pi r^3$, $\quad 1 - 2m/r = g^{-1}(dr, dr)$,

where spatial metrics are positive definite, in units $G = 1$.

Post-Newtonian expansion: for a perfect fluid,

$mc^2 = (\text{Newtonian mass})c^2 + \text{Newtonian kinetic energy} + \text{potential energy} \ + O(c^{-2})$.

$$\text{Large spheres: } m \to \left\{ \begin{array}{c} \text{Bondi} \\ \text{ADM} \end{array} \right\} \text{mass at } \left\{ \begin{array}{c} \text{null} \\ \text{spatial} \end{array} \right\} \text{infinity in asymptotically flat space-times.}$$

Small spheres: $m/\text{volume} \to$ density at a regular centre.

Trapping: a surface is $\left\{\begin{array}{c}\text{trapped}\\ \text{marginal}\\ \text{untrapped}\end{array}\right\}$ if $\left\{\begin{array}{c} r<2m \\ r=2m \\ r>2m \end{array}\right\}$.

Surface gravity $\kappa = *d*dr/2$, where d is the exterior derivative in the normal space, i.e. $*d*d$ is the wave operator in the normal space [Hayward 1998].

Then $k^a\nabla_{[b}k_{a]} \cong \pm\kappa k_b$, where $\cong$ denotes evaluation on a trapping horizon $r \cong 2m$, similarly to the usual Killing identity.

A trapping horizon is $\left\{\begin{array}{c}\text{outer}\\ \text{degenerate}\\ \text{inner}\end{array}\right\}$ if $\left\{\begin{array}{c} \kappa>0 \\ \kappa=0 \\ \kappa<0 \end{array}\right\}$.

Examples are provided by Reissner-Nordström solutions:

the $\left\{\begin{array}{c}\text{future}\\ \text{past}\end{array}\right\}$ trapping horizons are the Killing horizons of the $\left\{\begin{array}{c}\text{black}\\ \text{white}\end{array}\right\}$ hole, outer, degenerate or inner as appropriate.

The Einstein equation implies $\kappa = m/r^2 - 4\pi r w$, where the work density is $w = -\frac{1}{2}\text{tr}\, T$ and the trace is in the normal space.

In vacuo, $\kappa = m/r^2$, therefore reducing to the Newtonian surface gravity as $c^{-1} \to 0$.

Thus κ also provides a relativistic definition of the surface gravity of planets and stars.

Another invariant of T is the energy flux $\psi = T \cdot g^{-1}(dr) + w dr$.

The Einstein equation implies $dm = A\psi + w dV$, which was dubbed the unified first law [Hayward 1998], as it encodes first laws of both thermodynamics and black-hole dynamics.

Essentially, it expresses energy conservation, with the terms on the right-hand side being interpreted respectively as energy supply and work.

Any spherically symmetric metric can locally be written dual-null coordinates $x^{\pm}$: $ds^2 = r^2 d\Omega^2 - 2e^{2\varphi}dx^+dx^-$, where $d\Omega^2$ refers to the unit sphere and (r, φ) are functions of (x^+, x^-).

Coordinate expressions for $(k, j, m, \kappa, w, \psi)$ and the unified first law follow [Hayward 1998].

3. Wormhole mouths

A wormhole mouth is defined as a temporal outer trapping horizon [Hayward 1999].

It is temporal in order to be two-way traversable, while the outer condition $\kappa > 0$ is proposed as the generalization of the minimality condition.

Either $\partial_+ r$ or $\partial_- r$ must vanish on the mouth, and $\partial_+ r \cong 0$ will be assumed henceforth.

Introducing a generating vector ξ of the marginal surfaces composing the mouth, its defining property is $\xi \cdot d(\partial_+ r) \cong 0$.

Writing $\xi = \xi^+ \partial_+ + \xi^- \partial_-$, one has $\xi^+ > 0$, $\xi^- > 0$ for future-pointing ξ.

Then the above expands as $\xi^+ \partial_+ \partial_+ r + \xi^- \partial_- \partial_+ r \cong 0$, then $\kappa = -e^{-2\varphi} \partial_+ \partial_- r$ shows that $\partial_+ \partial_+ r > 0$, which expresses strict minimality of the sphere in the null hypersurface generated in the ∂_+ direction.

Strict minimality has been assumed for simplicity.

Note that minimality alone [Hochberg & Visser 1998] can select maximal surfaces, rather than minimal surfaces, in a time-symmetric hypersurface.

The following analysis will be of a single wormhole mouth, though it should be stressed that two-way traversability requires two mouths with opposite senses, i.e. marginal in opposite null directions, in mutual causal contact.

4. Static wormholes

Locally one can always introduce coordinates (t, r_*) defined by $\sqrt{2}x^\pm = t \pm r_*$.
Then the metric takes the form $ds^2 = r^2 d\Omega^2 + e^{2\varphi}(dr_*{}^2 - dt^2)$.

In a static case with static Killing vector ∂_t, so that (r, φ) are independent of t, transforming from r_* to r yields $ds^2 = r^2 d\Omega^2 + (1 - 2m/r)^{-1} dr^2 - e^{2\varphi} dt^2$, which is essentially the Morris-Thorne form of the metric, with their jargon "shape function" for $2m$ and "redshift function" for φ.
Here φ is a gravitational potential, reducing to the Newtonian potential in the Newtonian limit, if t reduces to Newtonian time.
The Morris-Thorne metric is singular at the wormhole throat $r \cong 2m$, so they used an embedding method to express minimality, which is unnecessary but equivalent [Visser 1995].
It is not recommended to generalize by inserting time-dependent or angular-dependent factors into a metric which is singular precisely at the object of interest.

In static cases, a wormhole mouth as defined above must be a double trapping horizon, $\partial_+ r \cong \partial_- r \cong 0$, since $\partial_\pm = \sqrt{2}(\partial_t \pm \partial_*)$ and $\partial_t r = 0$.
Then $\partial_* r \cong 0$, so the surface is extremal in the static hypersurface.
Since $\partial_t \partial_t r = 0$, one finds that $\kappa > 0$ implies that the surface is strictly minimal in the static hypersurface, $\partial_* \partial_* r > 0$.
A calculation shows that $2\kappa \cong (m - r\partial_r m)/r^2$, so that $\kappa > 0$ reduces to the "flaring-out" condition of Morris & Thorne.
Thus their embedding method correctly expresses strict minimality in static cases.

Another calculation shows that $\kappa \cong 2\pi r(\tau - \rho)$, where $\rho = -T_t^t$ is the energy density and $\tau = -T_r^r$ the radial tension.
This confirms that the weak energy condition must be violated: $\tau > \rho$, $\tau - \rho$ equalling surface gravity over circumference.

Note that κ is an invariant measure of the radial curvature or "flaring-out": $R^{\hat{r}}{}_{\hat{\theta}\hat{r}\hat{\theta}} = R^{\hat{r}}{}_{\hat{\phi}\hat{r}\hat{\phi}} \cong -2\kappa/r$, so that κ, like principal curvature, has units of inverse length.

5. Laws of wormhole dynamics

Proofs of the following are all straightforward calculations using the Einstein equation.

Negative energy density: the null energy condition is violated on a wormhole mouth.

This confirms that exotic/ghost/phantom matter is required even in dynamic cases.

Since a black hole may be locally defined by a future outer trapping horizon [Hayward 1994], this allows interconversion of black holes and traversable wormholes.

A future outer trapping horizon characterizes a black hole if achronal, equivalently $T_{++} \geq 0$, and a wormhole if temporal, equivalently $T_{++} < 0$.

Thus a traversable wormhole can collapse to a black hole if its negative-energy source fails, or if enough positive-energy matter or radiation is pumped in.

Conversely, a black hole can be converted into a traversable wormhole by beaming in enough negative-energy radiation.

Zeroth law: κ is constant on a static wormhole throat.

Second law: future, past or static wormhole mouths respectively have decreasing, increasing or constant area.

This is like the second law of black-hole dynamics [Hayward 1994], but with reversed sign, reflecting the causal character of the mouth, or equivalently, the reversed null energy condition.

It follows that a static wormhole is enlarged or reduced, respectively, by opening then closing a region of past or future trapped surfaces.

To enlarge, this can be done by beaming in negative energy, balanced by subsequent positive energy, while the opposite order would reduce the area.

For some matter models, this can be done in an apparently stable way, while others are unstable, leading either to collapse to a black hole as above, or to inflationary expansion.

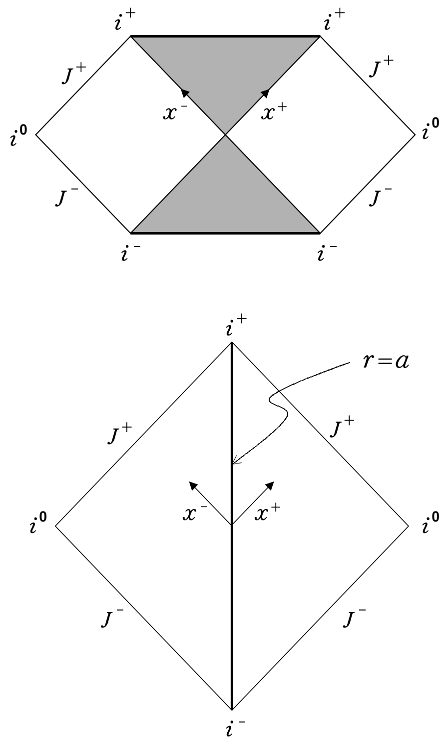
First law: $m' \cong \kappa A'/8\pi + wV'$, where the prime denotes $\xi \cdot d$.

This has the same form as the first law of black-hole dynamics [Hayward 1998].

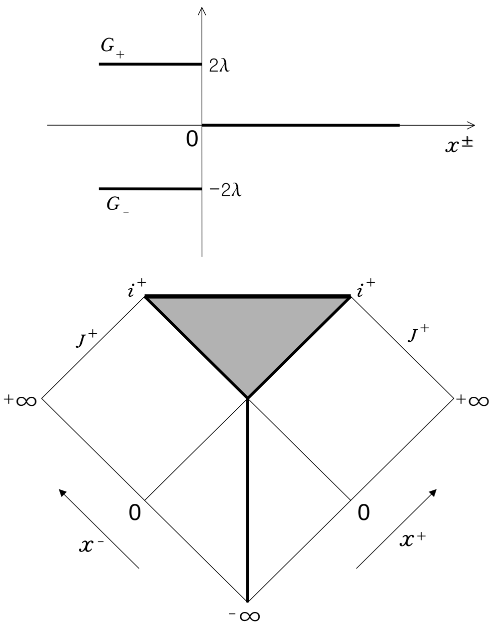
Temperature: an outer trapping horizon has a local Hawking temperature $\kappa/2\pi$ [Hayward et al. 2009, di Criscienzo et al. 2009], which therefore applies to wormhole mouths.

6. Examples

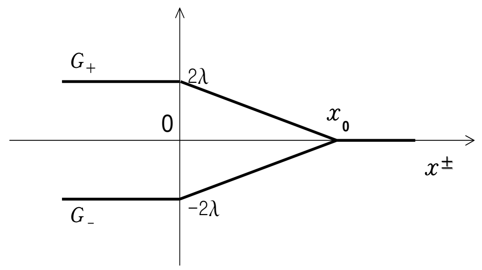
From S A Hayward, S-W Kim & H Lee, Phys. Rev. D65, 064003 (2002) / gr-qc/0110080.

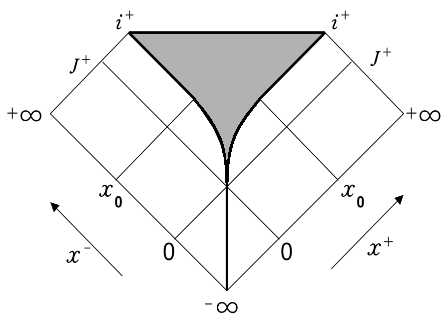


Penrose conformal diagrams of (a) the static black hole, (b) the static wormhole. Both space-times are divided into two universes (unshaded regions), but observers can travel freely between them via the wormhole, whereas the black hole (upper shaded region) swallows any such attempt.

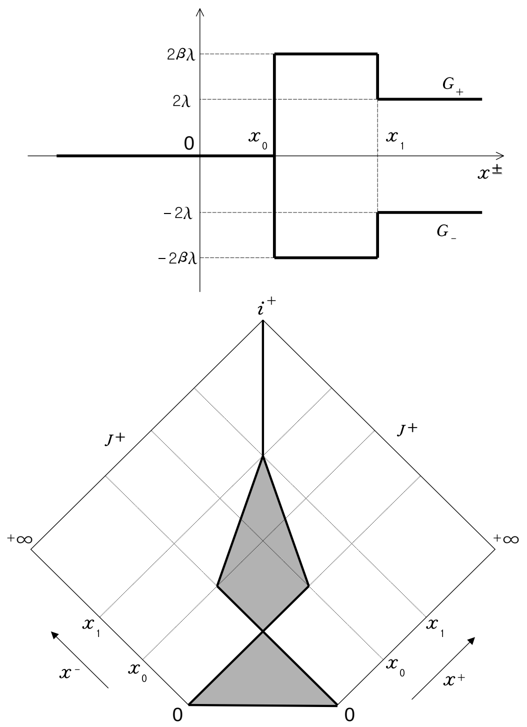


(a) The ghost field g is switched off suddenly from both sides of the wormhole. (b) Conformal diagram: when the wormhole's negative-energy source fails suddenly at $x^{\pm} = 0$, it immediately collapses into a black hole.

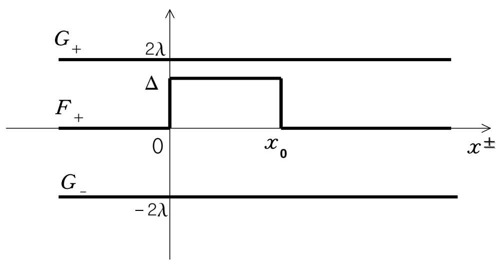


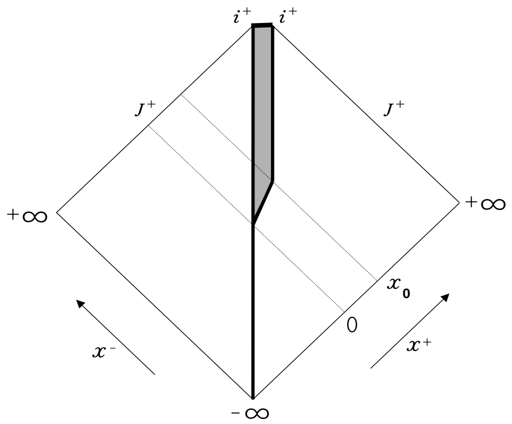


(a) The ghost field g is gradually reduced to zero. (b) Conformal diagram: the wormhole throat bifurcates and the resulting non-static wormhole again eventually becomes a black hole. Shading in these diagrams indicates trapped regions, where $\partial_+ r \partial_- r > 0$.

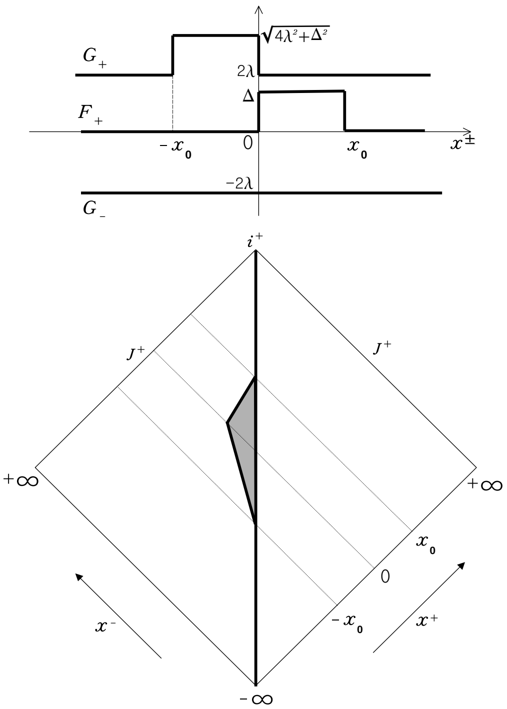


(a) Irradiating a vacuum black hole with the ghost field g. (b) Conformal diagram: the initially static black hole, becomes a dynamic wormhole, eventually reaching a static state. The black hole has been converted into a traversable wormhole.





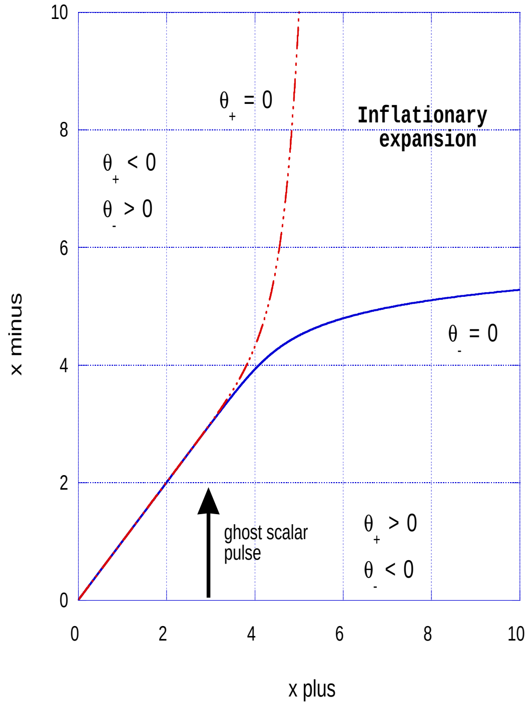
(a) A step pulse of positive-energy radiation is beamed through the wormhole. (b) Conformal diagram: the wormhole becomes non-static but, for a small-energy pulse, remains traversable for a long time.



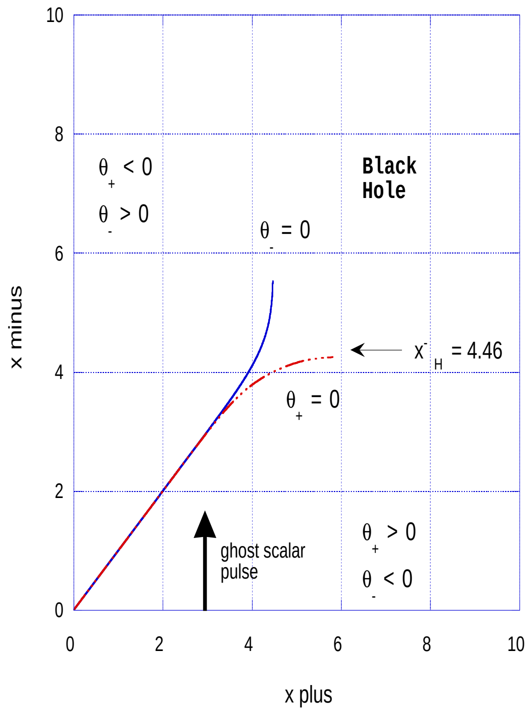
(a) The step pulse of positive-energy radiation is balanced by a preceding pulse of negative-energy radiation. (b) Conformal diagram: the wormhole returns to a static state.

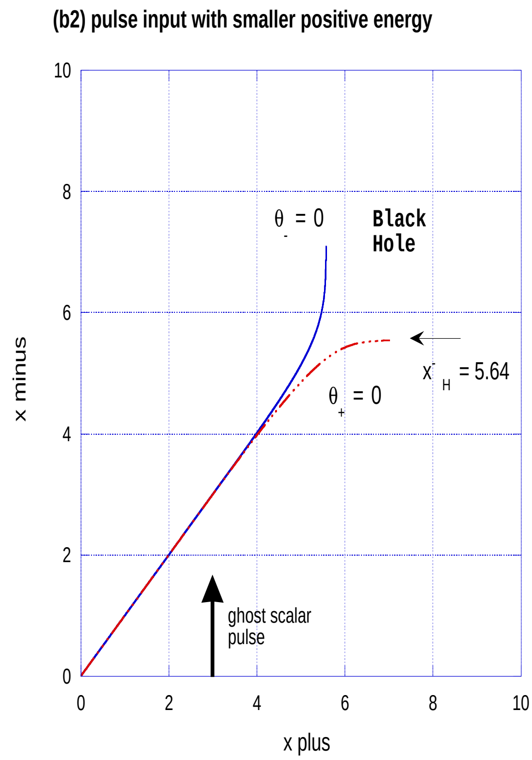
From H Shinkai & S A Hayward, Phys. Rev. D66, 044005 (2002) / gr-qc/0205041.

(a) pulse input with negative energy

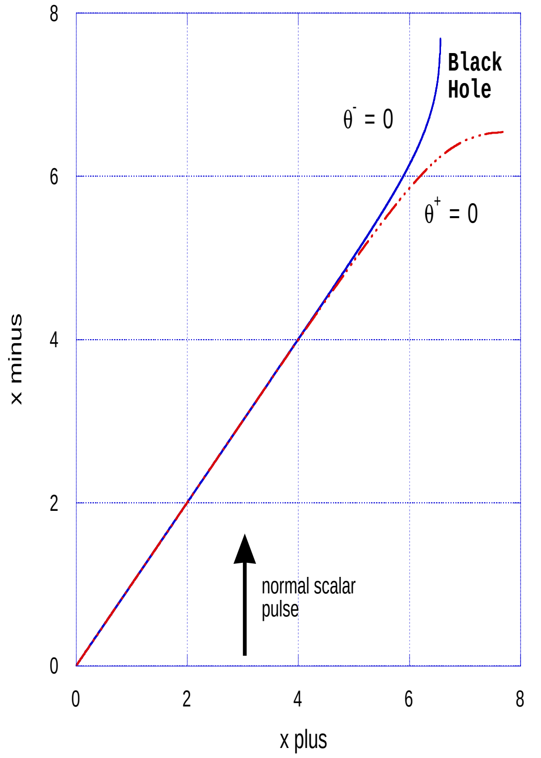


(b1) pulse input with positive energy

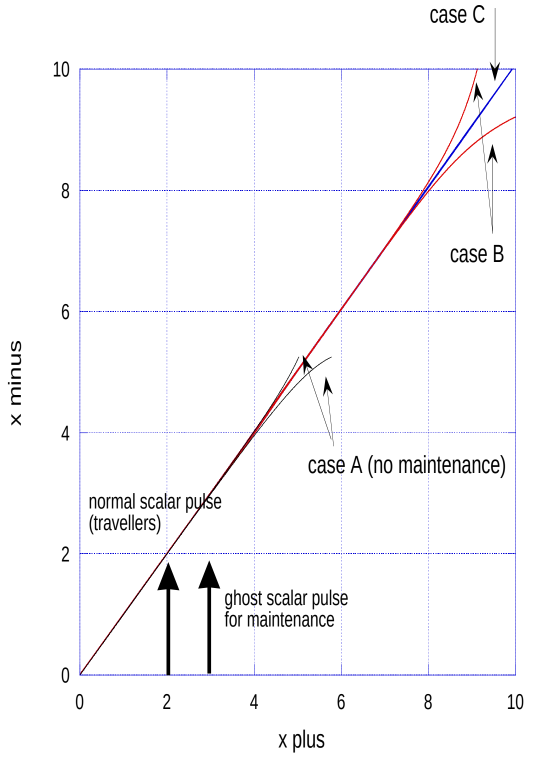




Horizon locations, $\vartheta_{\pm} = 0$, for perturbed wormhole. Fig.(a) is the case we supplement the ghost field, $c_a = 0.1$, and (b1) and (b2) are where we reduce the field, $c_a = -0.1$ and -0.01. Dashed lines and solid lines are $\vartheta_+ = 0$ and $\vartheta_- = 0$ respectively. In all cases, the pulse hits the wormhole throat at $(x^+, x^-) = (3, 3)$. A 45° counterclockwise rotation of the figure corresponds to a partial Penrose diagram.

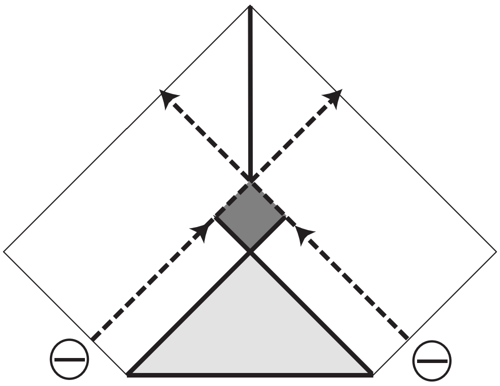


Evolution of a wormhole perturbed by a normal scalar field. Horizon locations: dashed lines and solid lines are $\vartheta_+ = 0$ and $\vartheta_- = 0$ respectively.

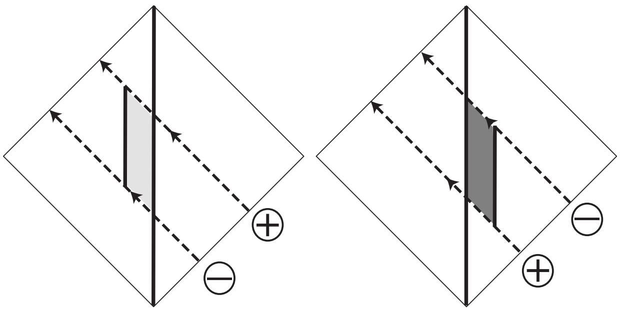


Temporary wormhole maintenance. After a normal scalar pulse representing a traveller, we beamed in an additional ghost pulse to extend the life of the wormhole. Horizon locations $\vartheta_+ = 0$ are plotted for three cases: (A) no maintenance, which results in a black hole; (B) with a maintenance pulse which results in an inflationary expansion; (C) with a more finely tuned maintenance pulse, which keeps the static structure up to the end of the range.

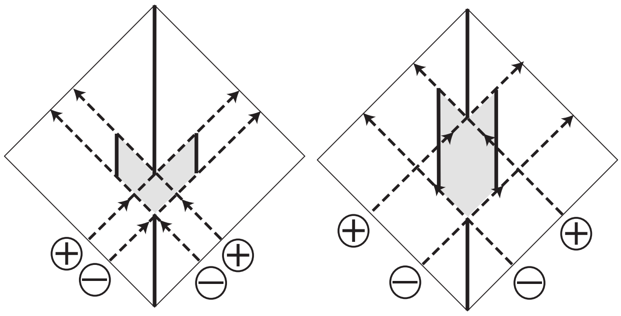
From H Koyama, S A Hayward & S-W Kim, Phys. Rev. D67, 084008 (2003) / gr-qc/0212106.



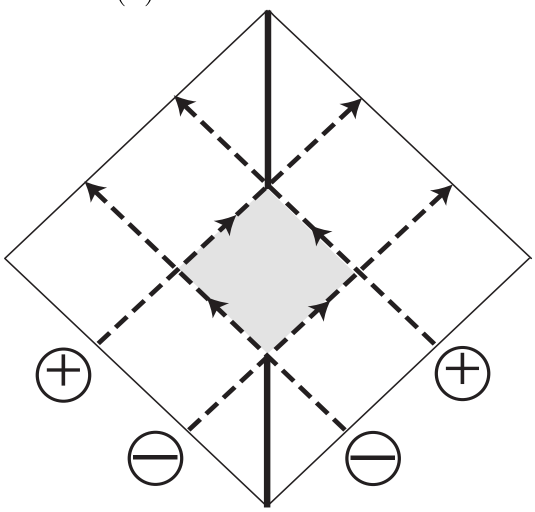
An HKL wormhole is constructed from a CGHS black hole by irradiating with impulsive negative-energy radiation, represented by dashed lines. The bold lines represent the trapping horizons, light shading indicates past trapped regions and darker shading indicates future trapped regions. The impulses shift the black-hole trapping horizons suddenly, making them coincide, then constant non-impulsive radiation supports the resulting wormhole.



An HKL wormhole is subjected to a double burst of impulsive radiation with equal positive and negative energy. The operation shifts one wormhole mouth away from, then back to, its original position. The shaded regions are (i) past or (ii) future trapped, respectively expanding or contracting, so the wormhole becomes respectively larger or smaller.

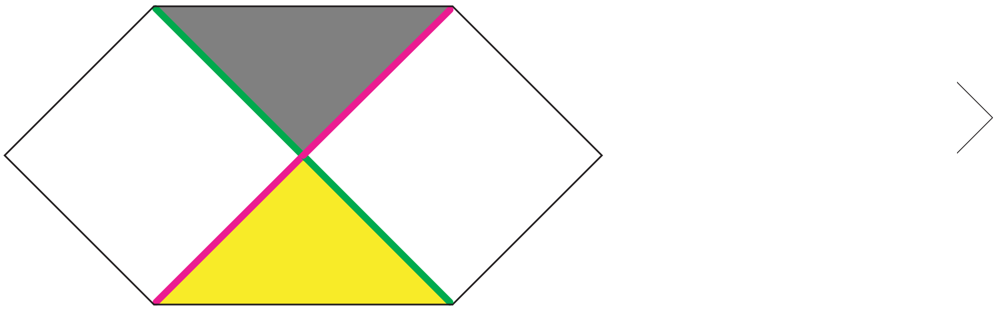


Wormhole enlargement by symmetric bursts of impulsive radiation, keeping the non-impulsive radiation constant. Both wormhole mouths are shifted out, then back again, opening and closing an expanding (past trapped) region, shaded for a rapid burst (i) and a slow burst (ii).

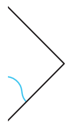


Wormhole enlargement process by symmetric bursts of impulsive radiation, with negative energy followed by positive energy, timed as described in the text. The non-impulsive radiation is switched off between the impulses. Then the middle shaded region is vacuum and expanding.

From H Koyama & S A Hayward, Phys. Rev. D70, 084001 (2004) / gr-qc/0406113,
S A Hayward & H Koyama, Phys. Rev. D70, 101502 (2004) / gr-qc/0406080.

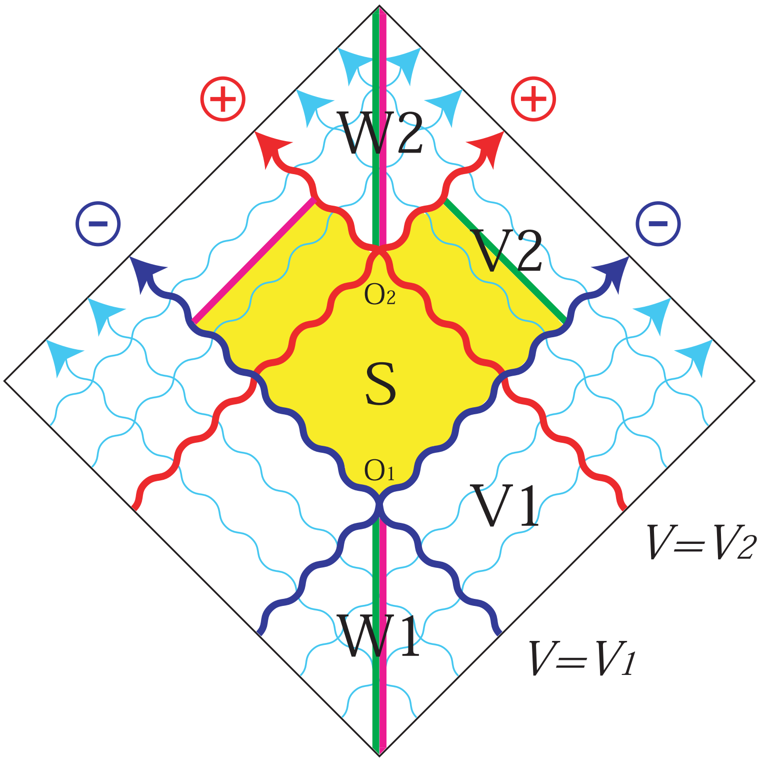


Penrose diagrams of a Schwarzschild black hole and a traversable wormhole. The bold magenta and green lines represent the trapping horizons, $\partial_+ A = 0$ and $\partial_- A = 0$, respectively. They constitute the Killing horizons of the black hole and the throat of the wormhole. Yellow and gray quadrants represent past trapped and future trapped regions, respectively. Wavy cyan lines represent the constant-profile phantom radiation supporting the wormhole structure.



V0

Penrose diagram of the wormhole construction model. The wavy blue lines represent impulsive radiation with negative energy density. The region S is Schwarzschild, V is Vaidya and W is static-wormhole.



Penrose diagram of the wormhole enlargement model. Wavy red lines represent impulsive radiation with positive energy density. The regions W1 and W2 are static-wormhole, V1 and V2 are Vaidya, and S is Schwarzschild.