Constant-expansion surfaces for finite-distance angular momentum estimates in numerical relativity

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• **Motivation:** estimating angular momentum in simulations of strongly gravitating systems;
  - For establishing balance laws, study transfer of angular momentum in non-vacuum scenarios;
  - For error control in numerical settings [Baker2007, Pollney2007];
  - For initial data [Caudill2006];

• **Background:**
  - Angular momentum ambiguities in General Relativity;
  - Measuring angular momentum in Numerical Relativity;
  - Constant-Expansion (CE) surfaces;

• **Results**

• **Conclusions**
Angular momentum ambiguities in General Relativity

Conservation laws for matter

- Noether procedure for the construction of conserved currents [Szabados2009 and references];
- In its possibly simplest form: test mass in free fall:

\[
\frac{du^a}{d\tau} + \Gamma^a_{bc} u^b u^c = 0
\]

\[
\xi_a \left( \frac{du^a}{d\tau} + \Gamma^a_{bc} u^b u^c \right) = \frac{d}{d\tau} (\xi_a u^a) - u^a u^b \xi_{a;b}
\]

- There is a conserved quantity for each Killing vector field:
  - In flat spacetimes, Poincaré group:
    - Translations → energy-momentum
    - Rotations → angular momentum
    - Lorentz boosts → center-of-mass conservation
  - In general spacetimes? [Harte2008]
**Angular momentum ambiguities in General Relativity**

**Conservation laws for the gravitational field**

- General Relativity: apply to the gravitational field?

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<th>Problems</th>
<th>Solutions</th>
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<tr>
<td>Localization: “local” energy-momentum and angular-momentum of the gravitational field are not observable (local flatness).</td>
<td>Global approach (spatial and null infinity)</td>
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<tr>
<td>Quasi-local approach (extended but finite regions)</td>
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<tr>
<td>Symmetry &amp; Conservation: in a generic gravitational field, which symmetries? How to define translations? Rotations?</td>
<td>Asymptotic symmetries</td>
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<tr>
<td>Quasi-local symmetries</td>
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<tr>
<td>Approximate symmetries</td>
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<tr>
<td>Origin: Choose an appropriate origin with respect to which angular momentum is calculated (center of mass problem in GR). Important if one wants to compare different spacetimes!</td>
<td>Good cuts, nice sections, CE surfaces?</td>
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</table>
In numerical simulations, however, it is not trivial to compute these quantities:
- Only finite-distance surfaces are usually available - how can one calculate the limit to spatial or null infinity?
- Gauge unknown, potentially non-trivial, not modifiable - how can one ensure that these corresponds to the correct chart/tetrad choice required by the global formalisms?
# Measuring Angular Momentum in Numerical Relativity

## Solutions

<table>
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<th>Approach</th>
<th>Solution</th>
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| Common approach in numerical simulations      | Use ADM angular momentum  
\[ J_{\text{ADM}} = \frac{1}{8} \int \varphi^a R^b K_{ab} d^2V \]  

or Bondi angular momentum  
\[ J_{\text{Bondi}}^i = \frac{1}{16\pi} \text{Re} \left\{ \sqrt{\int [\bar{m} (2\Psi_1 - 2\sigma \partial \sigma - \delta (\sigma \partial)) + \ell (\Psi_2 + \sigma \partial - \delta \sigma^2)] d^2V} \right\} \]  

with chart and tetrad build from simulation’s coordinates; extract on a set of constant coordinate radius surfaces and extrapolate to infinity, but using:  
\[ \Psi_4 = -\ddot{\sigma} \]  

| Gallo2008, Deadman2008                       | Use the transformation law to a Bondi chart/tetrad to rewrite the Bondi expressions at finite distance and in generic coordinates. |
| Our test                                     | Use the angular momentum from rotational Killing vector field  
\[ J_{\text{Killing}} = \frac{1}{8} \int \phi^a R^b K_{ab} d^2V \]  
on CE surfaces, which select a radial coordinate, a tetrad and a center of mass. |
In a 3+1 setting, given a 2-surface, a null tetrad can be constructed from the normal to the spatial hypersurfaces, the normal to the 2-surface on the spatial hypersurfaces and two vectors tangent to the 2-surface, using:

\[
\begin{align*}
\ell^a &= \frac{T^a + R^a}{\sqrt{2}} \\
n^a &= \frac{T^a - R^a}{\sqrt{2}} \\
m^a &= \frac{\theta^a + i\varphi^a}{\sqrt{2}}
\end{align*}
\]

The surface’s outgoing null expansion is defined as:

\[
\Theta(\ell) = q^{ab} \nabla_a \ell_b
\]

\[
q^{ab} = g^{ab} + \ell^a n^b + n^a \ell^b
\]

A 2-surface is said to be a CE surface if the expansion of its outgoing null normal is constant across it.

On simple, exact spacetimes, the expansion is asymptotically monotonic with the coordinate radius, and can be used as a radial coordinate;
Consider a Kerr spacetime in Kerr-Schild coordinates, and add a Lorentz boost and a translation:

\[ g_{ab} = \eta_{ab} + 2H k_a k_b \]
\[ H = \frac{Mr^3}{r^4 + a^2 z^2} \]
\[ k_a dx^a = -\frac{r(x dx + y dy) - a(x dy - y dx)}{r^2 + a^2} - \frac{dz}{r} - dt \]
\[ t \rightarrow \gamma(t - \beta y) \]
\[ x \rightarrow x + \Delta \]
\[ y \rightarrow \gamma(y - \beta t) \]

- Use a boost of 0.01668 and offset of (2, 4, 6, 8, 10)M.
- Construct two classes of surfaces: coordinate spheres and CE surfaces, located at areal radii of (2, 7.5, 15, 30, 60, 110)M;
- Integrate two types of angular momentum.
RESULTS

Coordinate spheres

$J_{ADM}^2 / M^2$ vs $R_{areal} / M$

- Red: $\Delta x = 0$
- Green: $\Delta x = 2M$
- Blue: $\Delta x = 4M$
- Magenta: $\Delta x = 6M$
- Cyan: $\Delta x = 8M$
- Yellow: $\Delta x = 10M$
RESULTS

Constant-Expansion surfaces

$J_{\text{TH}}/M^2$ vs $R_{\text{areal}}/M$

- $\Delta x = 0$
- $\Delta x = 2M$
- $\Delta x = 4M$
- $\Delta x = 6M$
- $\Delta x = 8M$
- $\Delta x = 10M$
RESULTS

Coordinate spheres

\[ \frac{J_{\text{HH}}^2}{M^2} \]

\[ R_{\text{areal}}/M \]

Legend:
- \( \Delta x = 0 \)
- \( \Delta x = 2M \)
- \( \Delta x = 4M \)
- \( \Delta x = 6M \)
- \( \Delta x = 8M \)
- \( \Delta x = 10M \)
RESULTS

Constant-Expansion surfaces

\[ \frac{J_{\text{ADM}}}{M^2} \]

\[ R_{\text{areal}}/M \]
RESULTS

\[
\gamma \beta \Delta x
\]

- \( J_{\text{ADM}}/M^2 - a/M, \text{CE} \)
- \( J_{\text{Killing}}/M^2 - a/M, \text{CE} \)
- \( J_{\text{ADM}}/M^2 - a/M, \text{CS} \)
- \( J_{\text{Killing}}/M^2 - a/M, \text{CS} \)

![Graph showing the relationship between \( \Delta x/M \) and \( \gamma \beta \Delta x \)]
CONCLUSIONS

• No natural way to resolve the supertranslation ambiguity in a generic spacetime - no corresponding structure in the asymptotic isometry group, ambiguity must be resolved by hand with a sensible prescription;
• Equivalent to the problem of finding a center of mass in a generic, asymptotically flat spacetime;
• Angular momentum constructions at null infinity will always contain one such (more or less explicit) assumption: if the assumption about the center of mass is not the intended one, the angular momentum will have a systematic error that does not vanish asymptotically;
• For (slightly) boosted Kerr, the Killing construction calculates the angular momentum with respect to an observer corresponding to the integration surfaces; the ADM integrand yields unclear results.