Hamiltonian of a spinning test-particle in curved spacetime

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Motivation/Content:

• Forthcoming GW observations ⇒ crucial to describe as accurately as possible the two-body dynamics in general relativity.

• Choosing a suitable spin supplementary condition, we derive the constrained Hamiltonian and obtain that the phase-space variables \( \{ x, P, S \} \) are canonical at linear order in the test-particle’s spin.
Describing the dynamics of spinning bodies

- Dynamics of spinning bodies in general relativity is rather complex. It has been investigated during the last seventy years starting from pioneering work by Mathisson 37, Papapetrou 51, Pirani 56, Tulczyjew 57 & Dixon 70.

- Spin effects on free motion of test particle first derived in the form of a coupling to the spacetime curvature. The calculation assumes test particle described by pole-dipole energy-momentum tensor ⇒ neglects quadrupole moment (and higher multipole moments) ⇒ provides spin couplings only at linear order in the test-particle’s spin.

- Two-body dynamics of spinning bodies also derived in a PN expansion.
  [Barker & O’Connell 75; Damour & Schafer 88, Kidder, Will & Wiseman 93; Kidder 95; Tagoshi et al. 01; Porto et al. 06-08; Faye et al. 06; Blanchet et al. 06; Steinhoff et al. 08; Hergt et al. 08; Damour et al. 08]
The Lagrangian and the Mathisson-Papapetrou-Pirani equations

- **Building on work by** Hanson & Regge 74 **on the relativistic spin-top dynamics in flat spacetime**, Porto 06 **recovered the Mathisson-Papapetrou-Pirani equations for a spinning particle in curved spacetime from the action**

\[ S = \int L(a_1, a_2, a_3, a_4) \, d\sigma \quad \sigma \rightarrow \text{parameter along a representative worldline} \]

- \( a_1, a_2, a_3, a_4 \rightarrow \) four Lorentz-invariant scalars depending on \( u^\mu \) and \( \Omega^{\mu\nu} \)

- \( u^\mu \equiv dx^\mu/d\sigma \rightarrow \) tangent vector to the representative worldline

- \( \Omega^{\mu\nu} = \eta^{AB} e^\mu_A \frac{D e^\nu_B}{D\sigma} \rightarrow \) describes how the tetrad carried by the test particle rotates along the worldline

- **Defining the four-momentum vector and spin tensor of the test particle as**

\[ p_\mu \equiv \frac{\partial L}{\partial u^\mu} \bigg|_\Omega, \quad S_{\mu\nu} \equiv 2 \frac{\partial L}{\partial \Omega^{\mu\nu}} \bigg|_u \quad \text{\( p_\mu \) is not the momentum conjugate to \( x^\mu \)!} \]

variations of the action with respect to \( e^\mu_A \) and the test particle’s position \( x^\mu \) give

\[ \frac{DS^{\mu\nu}}{D\sigma} = S^{\mu\lambda} \Omega^\nu_\lambda - \Omega^{\mu\lambda} S^\nu_\lambda = p^\mu u^\nu - p^\nu u^\mu, \quad \frac{Dp_\mu}{D\sigma} = -\frac{1}{2} R^\mu_{\alpha\beta\gamma} u^\alpha S^\beta\gamma \]
**Unconstrained Hamiltonian through a Legendre transformation**

- \( S = \int L \left(x^\mu, u^\mu, \phi^a, \frac{d\phi^a}{d\sigma}\right) d\sigma = \int L \left[x^\mu = (t, x^i), u^\mu = (1, v^i), \phi^a, \dot{\phi}^a\right] dt \)

\[ e^\mu_A(\phi, x) = \Lambda^B_A(\phi) \tilde{e}^\mu_B(x) \quad \Lambda^B_A \rightarrow \text{Lorentz transformation} \]

- **Considering the total variation of the Lagrangian as function of** \( x^i, v^i, \phi^a \) and \( \dot{\phi}^a \)

\[ \delta L = \frac{\partial L}{\partial x^i} \delta x^i + P_i \delta v^i + \frac{\partial L}{\partial \phi^a} \delta \phi^a + P_{\phi^a} \delta \dot{\phi}^a \]

and the total variation of the Lagrangian as function of \( x^i, v^i \) and \( \Omega^{\mu\nu} \), we get

\[ P_i = p_i + \frac{1}{2} \eta^{AB} S_{\mu\nu} \tilde{e}^\mu_A \tilde{e}^\nu_B; i, \quad P_{\phi^a} = \frac{1}{2} S_{\mu\nu} \Lambda^A_C \frac{\partial \Lambda_{CB}}{\partial \phi^a} \tilde{e}^\mu_A \tilde{e}^\nu_B \]

- **Applying a Legendre transformation**

\[ H = P_i v^i + P_{\phi^a} \dot{\phi}^a - L = -p_t - \frac{1}{2} \eta^{AB} S_{\alpha\beta} \tilde{e}^\alpha_A \tilde{e}^\beta_B; t \quad p_t = (\partial L)/(\partial u^0) \mid_\Omega \]

- **Phase-space algebra**

\[
\begin{align*}
\{ x^i, P_j \} & = \delta^i_j, \quad \{ x^i, x^j \} = 0 = \{ P_i, P_j \}, \quad \{ S^{\mu\nu}, P_i \} = S^{\alpha\beta} \tilde{e}^\alpha_A \tilde{e}^\beta_B; i + S^{\mu\nu} \tilde{e}^\mu_A \tilde{e}^\nu_B, \\
\{ S^{\mu\nu}, x^i \} & = 0, \quad \{ S^{\mu\nu}, S^{\alpha\beta} \} = S^{\mu\alpha} g^{\nu\beta} + S^{\nu\beta} g^{\mu\alpha} - S^{\mu\beta} g^{\nu\alpha} - S^{\nu\alpha} g^{\mu\beta}
\end{align*}
\]
Why the Hamiltonian is unconstrained

- Test-particle’s spin variables are encoded in the antisymmetric tensor $S^\mu\nu$, which a priori contains six degrees of freedom instead of three.

- To fix the unphysical degrees of freedom associated with the arbitrariness in the definition of $S^\mu\nu$, a choice must be made for the so-called spin supplementary condition (SSC).

For extended bodies, this arbitrariness can be interpreted, as the freedom in the choice for the location of the center-of-mass worldline of the body. [see e.g., Kidder 95; Semerak 99; Kyrian & Semerak 07]
How to deal with constraints in the Hamiltonian formalism

• Hamiltonian $H(q^i, \pi_i, t)$ ($i = 1, \ldots, n$) in $2n$-dimensional phase space; $\{\ldots, \ldots\} \Rightarrow$ antisymmetric and bilinear “bracket” operation satisfying Leibniz rule and Jacobi identity.

• For an arbitrary function $A$ we have: $\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, H\}$

• Consider a set of constraints $\xi_i = 0$, $i = 1, \ldots, 2m$ ($m < n$) such that the matrix $C_{ij} \equiv \{\xi_i, \xi_j\}$ is not singular.

• Replace original brackets with Dirac brackets which are the projection of the original symplectic structure onto the phase-space hypersurface defined by constraints, i.e.

$$\{A, B\}_{DB} = \{A, B\} + \{A, \xi_i\} [C^{-1}]_{ij} \{B, \xi_i\}$$

• The constrained Hamiltonian $\bar{H}$ is obtained by inserting the constraints in the original Hamiltonian $H$, and

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, \bar{H}\}_{DB}$$

[see Henneaux & Teitelboim 92]
Building the phase-space algebra in the Newton-Wigner SSC

- We generalize the Newton-Wigner SSC to curved spacetime as follows
  \[ V^\mu \equiv S_{\mu\nu}\omega^\nu = 0 \]
  \[ \omega_\mu = p_\mu - m \tilde{e}_\mu^T \]

- Moreover, three constraints must be imposed on the configuration coordinates \( \phi^a \)
  \[ \chi_\mu = \Lambda_T^A(\phi)(\tilde{e}_A)_\mu - \frac{p_\mu}{m} = 0 \]
  \[ m = \sqrt{-p_\mu p^\mu} \rightarrow \text{mass of test particle} \]

- Vector of constraints: \( \xi \equiv (V^1, V^2, V^3, \chi_1, \chi_2, \chi_3) \)
  \[ \{V^i, V^j\} = \omega_\mu S_{ij} \]
  \[ \{V^i, \chi_j\} = \frac{\omega_\mu p_\mu}{m} \left[ \delta^i_j - \frac{p^i \omega_j}{\omega_\mu p^\mu} \right] + S^i_\lambda T^T_{\lambda;\nu} \left[ \delta^\nu_j + \frac{p^\nu p_j}{m^2} \right] + O(S^2) \]
  \[ \{\chi_i, \chi_j\} = \frac{1}{2m^4} \left[ p_i R_{j\lambda\mu\nu} - p_j R_{i\lambda\mu\nu} \right] p^\lambda S_{\mu\nu} - \frac{1}{2m^2} R_{ij\mu\nu} S_{\mu\nu} + O(S^2) \]

We drop the remainders scaling as the square of the particle’s spin, because
the pole-dipole particle model is valid only at linear order in the particle’s spin.
Phase-space algebra in the generalized Newton-Wigner SSC

\[
K^{-1} = \begin{pmatrix} \mathbb{0}_3 & -(Q^{-1})^T \\ Q^{-1} & 0 \end{pmatrix} \quad [Q^{-1}]_{ij} = \frac{m}{\omega_{\mu\nu}^{\prime}} \left( \delta^i_j + \frac{\omega_{j\nu}^{\prime} p_i}{\omega_{\mu}^{\prime} p} \right)
\]

\[
C^{-1} = K^{-1} - K^{-1} \Sigma K^{-1} + \mathcal{O}(S^2)
\]

\[
\Sigma^{ij} \equiv \begin{pmatrix} \omega_{\mu}^{\prime} \omega_{\mu}^{\prime} S^{ij} \\ -S^{i\mu} \varepsilon^{\mu T}_{\mu;\nu} \left[ \delta^\nu_i + \frac{p_{\nu}^{\prime} p_i}{m^2} \right] \\ -\frac{1}{2m^2} R_{k\lambda \mu \nu} S^{\mu \nu} \left[ \delta^k_i \delta^\lambda_j + \frac{p_{\lambda}^{\prime} (\delta^k_i p_j - \delta^k_j p_i)}{m^2} \right] \\ -S^{j\mu} \varepsilon^{\mu T}_{\mu;\nu} \left[ \delta^\nu_j + \frac{p_{\nu}^{\prime} p_j}{m^2} \right] \\ -\frac{1}{2m^2} R_{k\lambda \mu \nu} S^{\mu \nu} \left[ \delta^k_j \delta^\lambda_i + \frac{p_{\lambda}^{\prime} (\delta^k_j p_i - \delta^k_i p_j)}{m^2} \right] \end{pmatrix}
\]

\[
\{x^i, x^j\}_{DB} = \left[ \frac{\omega_{\mu}^{\prime} \omega_{\mu}^{\prime} - 2p_{\nu}^{\prime} \omega_{\nu}^{\prime}}{(p_{\sigma}^{\prime} \omega_{\sigma}^{\prime})^2} \right] \left( S^{ij} - S^{i\mu} \frac{p_j}{p} + S^{j\mu} \frac{p_i}{p} \right) + \mathcal{O}(S^2) \quad \text{but } \omega_{\mu}^{\prime} \omega_{\mu}^{\prime} - 2p_{\nu}^{\prime} \omega_{\nu}^{\prime} = 0
\]

\[
\{x^i, P_j\}_{DB} = \delta^i_j + \mathcal{O}(S^2), \quad \{P_i, P_j\}_{DB} = \mathcal{O}(S^2), \quad S^{I} = S^{\mu \nu} \varepsilon^{I \mu}_{\mu} \varepsilon^{J \nu}_{\nu}, \quad S^{I} = \frac{1}{2} \varepsilon^{IJK} S^{JK}
\]

\[
\{x^i, S^J\}_{DB} = \mathcal{O}(S^2) = \{P_i, S^J\}_{DB}, \quad \{S^I, S^J\}_{DB} = \epsilon^{IJK} S^K + \mathcal{O}(S^2)
\]

The phase-space variables \(\{x^i, P_j, S^K\}\) provided by the generalized NW SSC are canonical at linear order in the particle’s spin.
Explicit Hamiltonian in the generalized Newton-Wigner SSC

- Dirac brackets of $m$ with $\{x^i, P_j, S^K\}$ are zero at linear order in the particle's spin.

- Using $m = \sqrt{-p_\mu p^\mu}$ we can write $p_t = -\beta^i p_i - \alpha \sqrt{m^2 + \gamma^{ij} p_i p_j}$

  where $\alpha = \frac{1}{\sqrt{-g_{tt}}}$, $\beta^i = \frac{g_{ti}}{g_{tt}}$, $\gamma^{ij} = g^{ij} - \frac{g_{ti} g_{tj}}{g_{tt}}$

- $\bar{H} = \beta^i p_i + \alpha \sqrt{m^2 + \gamma^{ij} p_i p_j} - F^K_t S^K + \mathcal{O}(S^2)$

  $F^K_\mu = \left(2 E_{\mu I J} \bar{\omega}_I + E_{\mu I J} \right) \epsilon^{IJK}$, $E_{\mu \alpha \beta} \equiv \frac{1}{2} \eta_{AB} \tilde{e}_A^\alpha \tilde{e}_B^\beta ; \mu$

  $\bar{\omega}_i = P_i - m \tilde{e}_i^T$, $\bar{\omega}_t = -\beta^i P_i - \alpha \sqrt{m^2 + \gamma^{ij} p_i p_j} - m \tilde{e}_t^T$

- Finally, $\bar{H}$ at linear order in the particle's spin, reads

  $\bar{H} = \beta^i P_i + \alpha \sqrt{m^2 + \gamma^{ij} p_i p_j} - \left[ \beta^i F^K_i + F^K_t + \frac{\alpha \gamma^{ij} P_i P_j^K}{\sqrt{m^2 + \gamma^{ij} p_i p_j}} \right] S^K$
Agreement with the ADM canonical Hamiltonian in PN theory

• Using the Kerr metric in ADM-TT coordinates [Hergt et al. 08] and the tetrad

\[ \tilde{e}_T^\mu = \delta_t^\mu \alpha, \quad \tilde{e}_I^\mu = \frac{\delta_I^\mu}{\sqrt{A}} + \mathcal{O}(8), \] denoting \( L^i \equiv \epsilon^{ijk} x^j P_k, \) \( \hat{P} = \frac{1}{m} P, \) \( S^* = \frac{M}{m} S \)

\[ \bar{H}_{1.5PN} = \frac{1}{r^3} \left( 2S_{Kerr} + \frac{3}{2} S^* \right) \cdot L, \quad \bar{H}_{2PN} = \frac{m}{2Mr^3} (3n_{ij} - \delta_{ij}) S_{Kerr}^i \left( S_{Kerr}^j + 2S_j^* \right) \] [Damour & Schaefer 88] [Damour 01]

\[ \bar{H}_{2.5PN} = \frac{1}{r^3} \left[ -\frac{M}{r} \left( 6S_{Kerr} + 5S^* \right) - \frac{5}{8} \hat{P}^2 S^* \right] \cdot L \] [Damour, Jaranowski & Schaefer 08]

\[ \bar{H}_{3PN} = \frac{m}{2Mr^3} S_{Kerr}^{ij} \left[ \frac{3}{2} \hat{P}^2 \left( 3n_{ij} - \delta_{ij} \right) - \frac{M}{r} \left( 9n_{ij} - 5\delta_{ij} \right) \right] + \frac{6m}{r^4} S^i S_{Kerr}^j (\delta_{ij} - 2n_{ij}) \]

\[ + \frac{3mn_{ij}}{2Mr^3} \left[ 2 \hat{P}^i S_{Kerr}^k \hat{P} [j S^{* k}] - (\hat{P} \times S^*)^i (\hat{P} \times S_{Kerr})^j \right] \] [Steinhoff et al. 08]

\[ \bar{H}_{3.5PN} = \frac{9m}{2M^2 r^4} (S_{Kerr} \cdot n) (S^* \times S_{Kerr}) \cdot \hat{P} - \frac{1}{4M^2 r^5} \left( 5(S_{Kerr} \cdot n)^2 - S_{Kerr}^2 \right) \times \]

\[ (9S^* + 4S_{Kerr}) \cdot L + \frac{21M^2}{2r^5} S_{Kerr} \cdot L + \left( \frac{7}{16r^3} \hat{P}^4 + \frac{27M^2}{8r^4} \hat{P}^2 + \frac{105M^2}{8r^5} \right) (S^* \cdot L) \]
Hamiltonian for a spinning test-particle in Schwarzschild

- Schwarzschild in isotropic coordinates:

\[ ds^2 = - \left( \frac{1-M}{1+M} \right)^2 dt^2 + (1+\frac{M}{2r})^4 (dx^2 + dy^2 + dz^2), \quad r^2 = x^2 + y^2 + z^2 \]

- Tetrad

\[ \tilde{e}^\mu_T = \left( \frac{1+M}{1-M} \right) \delta^\mu_0, \quad \tilde{e}^\mu_I = \left( 1+\frac{M}{2r} \right)^{-2} \delta^\mu_I \equiv \psi^2 \delta^\mu_I \]

- Hamiltonian

\[ \bar{H} = \bar{H}_{\text{nospin}} + \psi^6 \frac{1}{r^3} \left[ \frac{1-\frac{M}{2r} + \sqrt{Q} \left( 2-\frac{M}{2r} \right)}{\sqrt{Q} (1+\sqrt{Q})} \right] (L \cdot S^*) + O(S^2) \]

\[ Q = 1 + \psi^4 \hat{P}^2, \quad \hat{P}^2 = \delta^{JK} P_J P_K / m^2, \quad L \cdot S^* = r \epsilon^{IJK} n_I P_J \left( \frac{MS_K}{m} \right) \]
Conclusions

• We have generalized the Newton-Wigner SSC to curved spacetime.

• We have found that the phase-space variables $\{x^i, P_j, S^K\}$ provided by the generalized NW SSC are canonical at linear order in the particle’s spin.

• The constrained Hamiltonian agrees with the ADM canonical Hamiltonian computed in PN theory.

• Our method provides all higher-order PN couplings at linear order in the particle’s spin.

• We computed the explicit Hamiltonian in Schwarzschild and Kerr.

• We will employ the Hamiltonian for a spinning test-particle to build a new effective-one-body Hamiltonian for spinning bodies of comparable mass.