Deformations of space-time metrics

D.Pugliese[c],

[c] ICRA and Università di Roma “La Sapienza”

In collaboration with
S. Capozziello[a,b] , C. Stornaiolo[b]

[a] Dipartimento di Scienze Fisiche, Università di Napoli ”Federico II”,
And
[b] INFN, Sez. di Napoli,

•To adapt a known exact solution to a more realistic one ...

Solutions with simmetries

Schwarzschild 's metric

$$ds^2 = -\left(1 - \frac{2MG}{c^2 r}\right) dt^2 + \left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin\theta^2 d\phi^2\right)$$

FLRW Universe

$$ds^2 = c^2 dt^2 - a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2\right]$$

$$g = \Omega \eta$$

Deformations 1

$$(M, g)$$

$$g_{ab} = \eta_{AB}\omega^A_a\omega^B_b, \quad g^{ab} = \eta^{AB}e^a_A e^b_B,$$

Tensorial index

Tetrad index

Matrix of scalar field

$$\phi_B{}^A$$

$$\tilde{g}_{ab} \equiv \eta_{AB}\phi^A{}_C\phi^B{}_D\omega^C_a\omega^D_b$$

Deformed tensor

- *"First deforming matrix"*

$$\phi_B{}^A \qquad \omega^A_a\phi^B{}_A \equiv \tilde{\omega}^B_a \qquad \tilde{\omega}^A_a$$

$$\psi^A{}_B \qquad \tilde{e}^a_A \equiv \psi_A{}^B e^a_B; \qquad \tilde{e}^a_A$$

$$\psi^A{}_D \equiv \phi^{-1A}{}_D$$

$$\tilde{\omega}^A_a\tilde{e}^a_B = \delta^A_B, \qquad \tilde{\omega}^A_a\tilde{e}^b_A = \delta^b_a.$$

Deformations 2

$$\widetilde{g}_{ab} = \eta_{\mathrm{AB}}\phi^{\mathrm{A}}{}_{\mathrm{C}}\phi^{\mathrm{B}}{}_{\mathrm{D}}\omega^{\mathrm{C}}_{a}\omega^{\mathrm{D}}_{b}$$

$$\widetilde{g}^{ab} = \eta^{\mathrm{AB}}\psi_{\mathrm{A}}{}^{\mathrm{C}}\psi_{\mathrm{B}}{}^{\mathrm{D}}e^{a}_{\mathrm{C}}e^{b}_{\mathrm{D}}$$

$$\widetilde{g}_{ab} = \eta_{\mathrm{AB}}\widetilde{\omega}^{\mathrm{C}}_{a}\widetilde{\omega}^{\mathrm{D}}_{b}$$

$$\widetilde{g}^{ab} = \eta^{\mathrm{AB}}\tilde{e}^{a}_{\mathrm{C}}\tilde{e}^{b}_{\mathrm{D}}$$

- *"Second deforming matrix"*

$$\mathcal{G}_{\mathrm{CD}} \equiv \eta_{\mathrm{AB}}\phi^{\mathrm{A}}{}_{\mathrm{C}}\phi^{\mathrm{B}}{}_{\mathrm{D}}, \quad \mathrm{e} \quad \mathcal{G}^{\mathrm{CD}} \equiv \eta_{\mathrm{AB}}\psi_{\mathrm{A}}{}^{\mathrm{C}}\psi_{\mathrm{B}}{}^{\mathrm{D}}$$

$$\widetilde{g}_{ab} = \mathcal{G}_{\mathrm{CD}}\omega^{\mathrm{C}}_{a}\omega^{\mathrm{D}}_{b}, \qquad \widetilde{g}^{ab} = \mathcal{G}^{\mathrm{CD}}e^{a}_{\mathrm{C}}e^{b}_{\mathrm{D}},$$

Some features of deformations

- Commutation relations

$$[\widetilde{e}_{\mathrm{A}}, \widetilde{e}_{\mathrm{B}}]^b = \psi_{\mathrm{A}}{}^{\mathrm{S}} \psi_{\mathrm{B}}{}^{\mathrm{T}} [e_{\mathrm{S}}, e_{\mathrm{T}}]^b + 2\psi_{[\mathrm{A}}{}^{\mathrm{S}} e^a_{|\mathrm{S}} e^b_{\mathrm{T}} \left(\partial_{a|} \psi_{\mathrm{B}]}{}^{\mathrm{T}}\right)$$

- Identity

$$g_{ab} = \eta_{\mathrm{AB}} \omega_a^{\mathrm{A}} \omega_b^{\mathrm{B}}, \quad g_{ab} = \eta_{\mathrm{AB}} \omega_a'^{\mathrm{A}} \omega_b'^{\mathrm{B}} = \eta_{\mathrm{AB}} \Lambda^{\mathrm{A}}{}_{\mathrm{C}} \Lambda^{\mathrm{B}}{}_{\mathrm{D}} \omega_a^{\mathrm{C}} \omega_b^{\mathrm{D}}$$

$$\Lambda^{\mathrm{A}}{}_{\mathrm{B}} \phi^{\mathrm{B}}{}_{\mathrm{C}}$$

- Diffeomorfism

$$\widetilde{g}_{\alpha\beta} = g_{\mu\nu} \phi^{\mu}{}_{\alpha} \phi^{\nu}{}_{\beta}$$

$$\phi^a{}_b \equiv \phi^{\mathrm{A}}{}_{\mathrm{B}} \omega_b^{\mathrm{B}} e_{\mathrm{A}}^a$$

Deformations of tridimensional metrics

$$g = \sigma h + \epsilon \, \boldsymbol{s} \otimes \boldsymbol{s}$$

- **Deformed metric** — g
- **Costant curvature metric** — h
- **1--form** — $\boldsymbol{s}$

$$h_{ab} = \eta_{AB}\omega_a^A\omega_b^B ,$$
$$g_{ab} = \mathcal{G}_{CD}\omega_a^C\omega_b^D$$

$$\widetilde{g}_{ab} = \eta_{AB}\phi^A{}_C\phi^B{}_D\omega_a^C\omega_b^D$$

$$\phi^A{}_C = \sqrt{\sigma}\delta^A{}_C + \alpha s^A s_C$$

$$f = \frac{n(n-1)}{2} .$$

$$||\mathbf{s}||_h = \sigma ,$$

Some features of deformed metrics

Conformal deformations

$$\widehat{g} \equiv \Omega^2 g = \Omega^2 \eta_{AB} \omega^A \otimes \omega^B$$

$$\psi^C{}_A = \Omega \Lambda^C{}_A$$

$$\widetilde{g}_{ab} = g_{ab} + h_{ab}$$

$$\widetilde{g}^{ab} = g^{ab} + X^{ab}$$

Deformed metric

Tensor

$$X^{ab} = \mathcal{D}^{AB} e^a_A e^b_B$$

$$\mathcal{G}_{AB} = \eta_{AB} + \mathcal{C}_{AB}$$

$$\mathcal{G}^{AB} = \eta^{AB} + \mathcal{D}^{AB}$$

$$\mathcal{D}^{AD} = -\eta^{AB} \mathcal{G}^{CD} \mathcal{C}_{BC}$$

$$X^{ab} h_{bc} \neq \delta^a_c$$

$$X^{ad} = -g^{ab} \widetilde{g}^{cd} h_{bc}$$

$$\mathcal{D}^{AB} \mathcal{C}_{BC} \neq \delta^A{}_C$$

$$\tilde{g}_{ab} = g_{ab} + h_{ab}, \qquad \tilde{g}^{ab} = g^{ab} + X^{ab}$$

$$\mathcal{G}_{\mathrm{AB}} = \eta_{\mathrm{CD}} \phi^{\mathrm{C}}{}_{\mathrm{A}} \phi^{\mathrm{D}}{}_{\mathrm{B}}$$

$$\phi^{\mathrm{A}}{}_{\mathrm{B}} = \delta^{\mathrm{A}}{}_{\mathrm{B}} + \varphi^{\mathrm{A}}{}_{\mathrm{B}}$$

$$\varphi^{\mathrm{A}}{}_{\mathrm{A}} = 0, \quad \varphi^{\mathrm{A}}{}_{0} = \varphi^{0}{}_{\mathrm{A}}, \quad \varphi^{\mathrm{A}}{}_{\mathrm{i}} = -\varphi^{\mathrm{i}}{}_{\mathrm{A}} \quad \text{dove} \quad i \in \{1, 2, 3\}$$

$$\varphi^{\mathrm{B}}{}_{\mathrm{A}} = \begin{pmatrix} 0 & a & b & c \\ a & 0 & d_1 & d_2 \\ b & -d_1 & 0 & d_3 \\ c & -d_2 & -d_3 & 0 \end{pmatrix}$$

$$\begin{aligned} \tilde{g}_{ab} &= \Omega^2 g_{ab} + h_{ab} \\ \tilde{g}^{ab} &= \frac{1}{\Omega^2} g^{ab} + X^{ab} \end{aligned}$$

$$h_{ab} \equiv \eta_{\mathrm{AB}} \varphi^{\mathrm{A}}{}_{\mathrm{C}} \varphi^{\mathrm{B}}{}_{\mathrm{D}}\, \omega^{\mathrm{C}}_a \omega^{\mathrm{D}}_b, \qquad X^{ad} \equiv -\Omega^{-2} g^{ab}\, \tilde{g}^{cd}\, h_{bc}$$

$$\phi^{\mathrm{A}}{}_{\mathrm{B}} = \Omega \delta^{\mathrm{A}}{}_{\mathrm{B}} + \varphi^{\mathrm{A}}{}_{\mathrm{B}}\ ; \qquad \varphi = \begin{pmatrix} 0 & a & b & c \\ a & 0 & d_1 & d_2 \\ b & -d_1 & 0 & d_3 \\ c & -d_2 & -d_3 & 0 \end{pmatrix}$$

•Gravitational Waves

$$G_{ab} = 0$$

$$\tilde{g}_{ab} = g_{ab} + h_{ab}$$

$$h_{ab} = \mathcal{C}_{\mathrm{AB}}\omega^{\mathrm{A}}_{a}\omega^{\mathrm{B}}_{b}$$

$$\mathcal{G}_{\mathrm{AB}} = \eta_{\mathrm{AB}} + \mathcal{C}_{\mathrm{AB}}$$

$$|\,\mathcal{C}_{\mathrm{AB}}| \ll 1 \quad \forall \mathrm{A} \; e \; \mathrm{B},$$

$$\tilde{g}^{ab} = g^{ab} - h^{ab}$$

Gauge

$$\nabla^a h_{ab} = 0, \quad h = 0.$$

$$h \equiv g^{bd} h_{bd}$$

Equations

$$\Box h_{ac} = 2R^{b}{}_{ac}{}^{d} h_{bd}$$

$$(\Box \mathcal{C}_{\mathrm{AB}})\,\omega^{\mathrm{A}}{}_{a}\omega^{\mathrm{B}}{}_{c} = \mathcal{C}_{\mathrm{AB}}\left[2R^{b}{}_{ac}{}^{d}\omega^{\mathrm{A}}{}_{b}\omega^{\mathrm{B}}{}_{d} - \Box\left(\omega^{\mathrm{A}}{}_{a}\omega^{\mathrm{B}}{}_{c}\right)\right]$$

Deformations of Minkowski space time

$$\Box h_{ac} = 0$$

$$(\Box \mathcal{C}_{\mathrm{AB}})\,\omega^{\mathrm{A}}{}_{a}\omega^{\mathrm{B}}{}_{b} = 0$$

- Geodetic motions

$$v^a \nabla_a v^b = 0$$

Gravitational lensing

Geodetic deviations

Gravitational redshift

$$v^a \widetilde{\nabla}_a v^d = \frac{1}{2} \widetilde{g}^{cd} \{ \nabla_b \widetilde{g}_{ac} + \nabla_a \widetilde{g}_{bc} - \nabla_c \widetilde{g}_{ba} \} v^a v^b.$$

$$v^a \widetilde{\nabla}_a v^d = \frac{1}{2} \widetilde{g}^{cd} \{ \nabla_a h_{cb} + \nabla_b h_{ca} - \nabla_c h_{ba} \} v^a v^b$$

$$\widetilde{g}_{ab} = g_{ab} + h_{ab}, \qquad \widetilde{g}^{ab} = g^{ab} + X^{ab}$$

$$X^{dc}(h_{cb,a} + h_{ca,b} - h_{ba,c}) v^a v^b + X^{dc}(g_{cb,a} + g_{ca,b} - g_{ba}$$

$$+ g^{dc}(h_{cb,a} + h_{ca,b} - h_{ba,c}) v^a v^b = 0.$$

$$\begin{aligned} \widetilde{g}_{ab} &= \Omega^2 g_{ab} + h_{ab} \\ \widetilde{g}^{ab} &= \frac{1}{\Omega^2} g^{ab} + X^{ab} \end{aligned}$$

$$\begin{aligned} v^a \widetilde{C}^b{}_{ac} v^c &= 2(\ln \Omega)_{,v} v^b - (g_{ac} v^a v^c) \left[g^{bd} \nabla_d \ln \Omega + X^{bd} \nabla_d \ln \Omega \right] + \\ &+ 2 v^a (\nabla_a \ln \Omega) g_{dc} X^{db} v^c + \frac{1}{2} v^a \left[(\nabla_a h_{cd}) + (\nabla_c h_{ad}) + \right. \\ &- \left. (\nabla_d h_{ac}) \right] v^c \left(g^{bd} \Omega^{-2} + X^{bd} \right) \end{aligned}$$

- Gravitational lensing

$$\frac{dp^y}{d\lambda} = -\widetilde{\Gamma}^y{}_{\beta\gamma} p^\beta p^\gamma$$

$$\Delta p^y \equiv -\int_{-\infty}^{+\infty} \Gamma^y{}_{\beta\gamma} p^\beta p^\gamma \, d\lambda,$$

$$\delta p^y = -\frac{1}{2}\int_{-\infty}^{+\infty} \widetilde{g}^{y\delta} \left[\partial_\beta \left(\mathcal{G}_{\mathrm{CD}}\, \omega_\gamma^{\mathrm{C}} \omega_\delta^{\mathrm{D}}\right) + \partial_\gamma \left(\mathcal{G}_{\mathrm{CD}}\, \omega_\beta^{\mathrm{C}} \omega_\delta^{\mathrm{D}}\right) - \partial_\delta \left(\mathcal{G}_{\mathrm{CD}}\, \omega_\beta^{\mathrm{C}} \omega_\gamma^{\mathrm{D}}\right)\right] p^\beta p^\gamma \, d\lambda,$$

$$\delta p^y = -\frac{1}{2}\eta_{\mathrm{QP}} \int_{-\infty}^{+\infty} \tilde{g}^{y\delta} \left[\partial_\beta \left(\tilde{\omega}_\gamma^{\mathrm{Q}} \tilde{\omega}_\delta^{\mathrm{P}}\right) + \partial_\gamma \left(\tilde{\omega}_\beta^{\mathrm{Q}} \tilde{\omega}_\delta^{\mathrm{P}}\right) - \partial_\delta \left(\tilde{\omega}_\beta^{\mathrm{Q}} \tilde{\omega}_\gamma^{\mathrm{P}}\right)\right] p^\beta p^\gamma \, d\lambda \, .$$

$$\delta p^y = \Delta p^y - \frac{1}{2}\int_{-\infty}^{+\infty} [X^{y\delta}(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}) + g^{y\delta}(h_{\delta\beta,\gamma} + h_{\delta\gamma,\beta} - h_{\beta\gamma,\delta})] p^\beta p^\gamma \, d\lambda +$$

$$-\frac{1}{2}\int_{-\infty}^{+\infty} X^{y\delta}(h_{\delta\beta,\gamma} + h_{\delta\gamma,\beta} - h_{\beta\gamma,\delta}) p^\beta p^\gamma \, d\lambda$$

- Gravitational lensing

$$\frac{d\mathbf{e}}{dl_{eucl}} = -\frac{2}{c^2}\nabla_\perp \Phi$$

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) \delta_{ij} dx^i dx^j$$

$$\nabla_\perp \Phi \;\equiv\; [\nabla\Phi - \mathbf{e}(\mathbf{e}.\nabla\Phi)]$$

$$\tilde{ds}^2 = A(t)\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - A(t)\left(1 - \frac{2\Phi}{c^2}\right) \delta_{ij} dx^i dx^j$$

$$\frac{d\mathbf{e}}{dl_{eucl}} = -\frac{2}{c^2}\nabla_\perp \Phi - (\partial_0 \ln A)\,\mathbf{e}$$

$$\tilde{ds}^2 = A(|x|)\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - A(|x|)\left(1 - \frac{2\Phi}{c^2}\right) \delta_{ij} dx^i dx^j$$

$$\frac{d\mathbf{e}}{dl_{eucl}} = -\nabla_\perp \hat{\Phi}$$

$$\hat{\Phi} \equiv \frac{2}{c^2}\Phi + \ln A$$

$$|x|^2 = x^2 + y^2 + z^2$$

- Geodetic deviations

$$a^a = T^c \widetilde{\nabla}_c v^a$$

$$a^a = -\widetilde{R}_{cbd}{}^a S^b T^c T^d.$$

$$a_0^a \equiv -R_{cbd}{}^a S^b T^c T^d$$

$$a^a = a_0^a - \overline{R}_{cbd}{}^a S^b T^c T^d$$

$$a^a = -2(\partial_{[b} \widetilde{C}^a{}_{|d|c]} + \widetilde{C}^f{}_{[c|d|} \widetilde{\Gamma}^a{}_{b]f} + \Gamma^f{}_{[c|d|} \widetilde{C}^a{}_{b]f}) S^b T^c T^d$$

- Gravitational redshift

$$z \equiv \frac{\lambda_2 - \lambda_1}{\lambda_1}$$

$$z = \frac{\nu_1}{\nu_2} - 1 = \frac{\sqrt{-\underline{\mathcal{G}_{\mathrm{AB}}} \omega_0^{\mathrm{A}} \omega_0^{\mathrm{B}}}\,|_{P_2}}{\sqrt{-\underline{\mathcal{G}_{\mathrm{AB}}} \omega_0^{\mathrm{A}} \omega_0^{\mathrm{B}}}\,|_{P_1}} - 1$$

Deformations of space time metrics

Conclusion

We have seen it is possible to deforme space--time metrics using a matrix of scalar fields defined on the manifold.

Deformations could be a generalization of the relation $g = \Omega\eta$

Deformations of space-time metrics could be a method to "adapt" a model solution to a more realistic one.

Deformations could be a way to to study the inverse problem in general relativity.

Deformations of space time metrics

- D. Pugliese, Deformazioni di metriche spaziotemporali, tesi di laurea quadriennale 2006-2007, Supervisors :S. Capozziello and C. Stornaiolo
- S. Capozziello and C. Stornaiolo, Space-Time deformations as extended conformal Transformations International Journal of Geometric Methods in Modern Physics 5, 185-196 (2008).
- B. Coll, A universal law of gravitational deformation for general relativity, Proc. of the Spanish Relativistic Meeting,EREs, Salamanca Spain (1998).
- B. Coll, J. Llosa and D. Soler, Three-dimensional metrics as deformations of a constant curvature metric, Gen. Rel. Grav.34 269 (2002).
- J. Llosa and D. Soler, On the degrees of freedom of a semi-Riemannian metric, Class. Quant. Grav. 22, 893 (2005)