Abstract: This talk is devoted to introduce a gauge theory of the Lorentz Group based on the ambiguity emerging in dealing with isometric diffeomorphism-induced Lorentz transformations. The behaviors under local transformations of fermion fields and spin connections (assumed to be coordinate vectors) are analyzed in flat space-time and the role of the torsion field within the generalization to curved space-time is briefly discussed. The fermion dynamics including the new gauge field is then analyzed assuming time-gauge and stationary solutions in the non-relativistic limit are founded.
Outline:

1. Internal space-time symmetries: the standard approach to a Lorentz Gauge Theory
2. Diffeomorphism induced Lorentz transformations and new connections
3. Formulation of the theory on flat space-time
4. Fermion dynamics in the non-relativistic limit: a generalized Pauli Equation
5. Generalization to curved space-time and the role of the torsion field
Internal space-time symmetries: the standard approach to a Lorentz Gauge Theory

Internal symmetries of the space-time: we focus on the description of GR as a gauge model (underlying the ambiguity that arises from this approach).

Tetrad formalism ($e_\mu^{\ a}$) for the local Minkowskian tangent space-time can recover the Lorentz symmetry

→ tetrad changes are defined as local Lorentz $trs$ between inertial references

\[
g_{\mu\nu} = \eta_{ab} \ e_\mu^{\ a} \ e_\nu^{\ b} \quad \quad e_\mu^{\ a} \ e_\mu^{\ b} = \delta^a_b \quad \quad e_\mu^{\ a} \ e_\nu^{\ a} = \delta^\nu_\mu
\]

$\mu = 0, 1, 2, 3$  coordinate indices  \quad $a = 0, 1, 2, 3$  Lorentz indices

\[
e_\mu^{\ a} \rightarrow \Lambda_b^a e_\mu^{\ b}
\]
Local Lorentz invariance of the scheme  $\rightarrow$ Covariant derivative

$$\partial_a \psi = e_a^\mu \partial_\mu \psi \text{ (coordinate scalar)} \quad \rightarrow \quad D_a \psi = (\partial_a + \Gamma_a^{(L)}) \psi \text{ (Lorentz vector)}$$

**Lorentz connections** $\omega_{\mu}^{ab}$:

$$\Gamma_a^{(L)} = e_\mu^\alpha \Gamma_\mu^{(L)} \quad \Gamma_\mu^{(L)} = \frac{1}{2} \omega_{\mu}^{cd} \Sigma_{cd} \quad \begin{array}{l}
\omega_{\mu}^{cd} = e^{cv} \nabla_\mu e^d_v = e_\mu^a \gamma_a^{cd}
\end{array}$$

$\Sigma_{cd}$: generators of the Lorentz Group (LG)  $\nabla_\mu$: coordinate covariant derivative

- Connections are called **spin connections**:
  $\rightarrow$ they restore the correct Dirac algebra in curved space-time.
  $\rightarrow$ the correct treatment of spinors leads to the introduction of those connections which guarantee a suitable gauge model for the Lorentz group even on flat space-time.
  $\rightarrow$ spinors are a particular representation of the LG.
This picture suggests *in appearance* the description of GR as a *gauge model*

Lorentz connections are the projected *Ricci rotation coefficients* \( \omega^a_{\mu} = e^c_\mu \gamma^a_{\ c} \):

\[
R^a_{\mu\nu} = \partial_\nu \omega^a_{\mu} - \partial_\mu \omega^a_{\nu} + \mathcal{F}^a_{\ cd\ ef} \omega^c_{\mu} \omega^d_{\nu} \omega^e_{\ f}
\]

\( \mathcal{F}^a_{\ cd\ ef} \): LG structure constants

The Hilbert-Einstein action for GR can be written in the form

\[
S(e, \omega) = -\frac{1}{4} \int e \, d^4 x \ e^a_\mu e^\nu_b R^{ab}_{\mu\nu}
\]

→ Variation wrt connections leads to the *II Cartan structure equation*

\[
\partial_\mu e^a_\nu - \partial_\nu e^a_\mu - \omega^a_{\mu} e^a_\nu b + \omega^a_{\nu} e^a_{\mu b} = 0
\]

→ Variation wrt tetrads gives the Einstein equations
In the usual approach, $\omega^{ab}_\mu$ transform like Lorentz gauge vectors under infinitesimal local Lorentz $trs$ - $\epsilon^{ab}$: infinitesimal parameter $\rightarrow \Lambda^b_a = \delta^b_a + \epsilon^b_a$

$$
\omega^{ab}_\mu \rightarrow \omega^{ab}_\mu - \partial_\mu \epsilon^{ab} + \frac{1}{4} F^{ab}_{cd} \epsilon^{cd} \omega^{ef}_\nu
$$

Riemann tensor is preserved by such a change
(in flat space-time, we deal with non-zero gauge connections, but a vanishing curvature)

**Ambiguity:** $\omega^{ab}_\mu$ exhibit the right behavior to play the role of Lorentz gauge fields, and GR assume the features of a gauge theory.

But:

$\rightarrow$ spin connections can be uniquely determined as functions of tetrads fields in terms of the Ricci rotation coefficients: non fundamental gauge fields

$\rightarrow$ Tetrads fields (Principle of General Covariance): two dependent dof
Diffeomorphism induced Lorentz transformations and new connections

The introduction of fermions into the dynamics requires to treat local Lorentz \( trs \) as the real independent gauge of GR. Because of the spinor behavior, it is crucial to investigate if diffeomorphisms can be reinterpreted as local Lorentz transformations.

Transformation laws:

\[
\omega_{\mu}^{ab} \rightarrow \text{Gauge potential under Lorentz } trs \text{ - (Lorentz indices)}
\]

\[
\omega_{\mu} \rightarrow \text{Coordinate vector under diffeomorphism - (Coordinate indices)}
\]

\[
\psi \rightarrow \text{Representation of the LG - } \psi(x) \rightarrow \psi'(x') = S \psi(x) \text{ (tangent bundle)}
\]

\[
\psi \rightarrow \text{Coordinate scalar - (no world indices)}
\]

If the two \( trs \) overlap: inconsistency - what is the nature of \( \omega_{\mu}^{ab} \) and \( \psi \)?
An isometric diffeomorphism induces orthonormal transformed basis $e_\mu^a$:
in this sense an isometry generates a local Lorentz transformation of the basis.

- **Infinitesimal isometric diffeomorphism:**

  $x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu (x) \quad \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \quad \text{(isometry condition)}$

  
  $e_\mu^a (x) \xrightarrow{D} e_\mu^a (x) + e_\rho^a (x) \frac{\partial \xi^\rho}{\partial x'^\mu}$

- **Infinitesimal Lorentz transformation:**

  $\Lambda^b_a (x) = \delta^b_a + \epsilon^b_a \quad e_\mu^a (x) \xrightarrow{L} e_\mu^a (x) + e_\mu^b (x) \epsilon^a_b$

The two *trs* overlap if:  

$\epsilon_{ab} = D_{[a} \xi_{b]} - R_{abc} \xi^c$

Isometry condition $\nabla_{(\mu} \xi_{\nu)} = 0$ must hold in order to have $\epsilon_{ab} = -\epsilon_{ba}$. 
If isometric diffeomorphism are allowed:

**Diffeomorphism induced Lorentz transformation**: a new gauge field $A_{\mu}^{ab} \neq \omega_{\mu}^{ab}$ must be introduced to restore the Lorentz invariance

→ *Flat space-time*: in the case $e_{\mu}^{a} = \delta_{\mu}^{a}$ spin connections vanish and they remain identically zero under diffeomorphisms.

Coordinate transformations = Lorentz rotations (gauge transformations):

$\omega_{\mu}^{ab}$ are inappropriate to restore local Lorentz invariance.

→ *Curved space-time*: $\omega_{\mu}^{ab}$ are assumed to behave like tensors under diffeomorphism induced rotations.

**obs.** If $\omega_{\mu}^{ab}$ behave like gauge vectors, the standard approach can be recovered (ambiguity of tetrads dependence)

→ *Arbitrary choice*

**obs.** Spinor $\psi$ can noway be a Lorentz scalar
Section 3

Formulation of the theory on flat space-time

Construction of a diffeo-induced Lorentz gauge model on a Minkowski space.

Riemann curvature tensor vanishes: spin connections $\omega_{\mu}^{ab}$ can be set to zero
(in general, they are allowed to be non-vanishing quantity in view of local Lorentz invariance)

$\rightarrow$ The introduction of Lorentz connections $A_{\mu}^{ab}$ as the gauge field of local LG on flat space-time (as far as the correspondence between an infinitesimal diffeomorphism and a local Local rotation is recovered)

The metric tensor can be expressed as: $g_{\mu\nu} = \eta_{ab}e_{\mu}^{a}e_{\nu}^{b}$

\begin{align*}
\text{Infinitesimal diffeomorphism} & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
obs. If vector fields are treated no inconsistency arises if the two trs overlap.

If \( \epsilon_a^b \equiv \partial^b \xi_a(x^c) \) the two transformation laws are the same

\[
D V_a'(x'^c) = V_a(x^c) + \partial_a \xi^b(x^c) V_b(x^c) \quad \quad L V_a'(x'^c) = V_a(x^c) + \epsilon_a^b V_b(x^c)
\]

and the LG loses its status of independent gauge group. To restore the proper number of degrees of freedom of a Lorentz tr, out of that of generic diffeo., the isometry condition \( \partial_b \xi_a + \partial_a \xi_b = 0 \) has to be imposed.

Spin-1/2 fields are described by the Lagrangian density on a 4D flat manifold

\[
\mathcal{L} = \frac{i}{2} \bar{\psi} \gamma^a \epsilon^\mu_a \partial_\mu \psi - \frac{i}{2} \bar{\psi} \gamma^a \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi
\]

which is invariant under global Lorentz trs
Formulation of the theory on flat space-time

Let us introduce a local Lorentz transformation:

\[ S = S(\Lambda(x)) \]

\[ \psi(x) \rightarrow S \psi(x) \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x)S^{-1} \]

where \( S \) is in every point a non-singular matrix.

**Infinitesimal transformations**

\[ \epsilon^a_b (x) \ll 1 \quad (\epsilon^{ab} = -\epsilon^{ba}) \quad \rightarrow \quad \Lambda^a_b = \delta^a_b + \epsilon^a_b \]

\[ S = I - \frac{i}{4} \epsilon^{ab} \Sigma_{ab} \quad \Sigma_{ab} = -\frac{1}{2} [\gamma_a, \gamma_b] \quad [\Sigma_{cd}, \Sigma_{ef}] = iF_{cdef}^{\quad ab} \Sigma_{ab} \]

- A spinor can not be a Lorentz scalar → for assumption the connections \( \omega_{\mu}^{ab} \) do not follows Lorentz gauge transformations.

New connections have to be introduced to restore Lorentz invariance

(Differently from vector fields, spinors have to recognize the isometric components of the diffeomorphism as a local Lorentz transformation, if accelerated coordinates are taken into account)
Formulation of the theory on flat space-time

Let us assume that $\gamma$ matrices transform like Lorentz vectors:

$$ S \gamma^a S^{-1} = (\Lambda^{-1})^a_b \gamma^b $$

- In particular, if $\omega_{\mu}^{ab} = 0$ they still vanish under a Lorentz gauge transformation which now can be seen as a diffeomorphism.

This way $\mathcal{L}$ invariance is restored by the new covariant derivative:

$$ D_\mu \psi = (\partial_\mu - i g A_\mu) \psi = (\partial_\mu - i g A^{ab}_\mu \Sigma_{ab}) \psi $$

The gauge invariance $\gamma^a e^\mu_a D_\mu \psi \rightarrow S \gamma^a e^\mu_a D_\mu \psi$ is provided by the gauge field:

$$ A_\mu = A^{ab}_\mu \Sigma_{ab} \neq \omega_{\mu}^{ab} $$

which transforms like:

$$ A_\mu \rightarrow S A_\mu S^{-1} - 4i S \partial_\mu S^{-1} $$

$$ A^{ab}_\mu \rightarrow A^{ab}_\mu - \partial_\mu \epsilon^{ab} + 4 \mathcal{F}^{ab}_{cdef} \epsilon^{ef} A^{cd}_\mu $$

(i.e.) as a natural Yang-Mill field associated to the Lorentz gauge Group living in the tangent bundle.
Fermion dynamics in the non-relativistic limit: a generalized Pauli Equation

For the 4-spinor $\psi$, the implementation of the *local Lorentz symmetry* ($\partial_\mu \to D_\mu$), in a flat space, leads to the Lagrangian density:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$$

$$\mathcal{L}_0 = \frac{i}{2} \bar{\psi} \gamma^a e^\mu_a \partial_\mu \psi - \frac{i}{2} e^\mu_a \partial_\mu \bar{\psi} \gamma^a \psi - m \bar{\psi} \psi$$

$$\mathcal{L}_{\text{int}} = \frac{1}{8} e^\mu_c \bar{\psi} \{ \gamma^c, \Sigma_{ab} \} A_{\mu}^{ab} \psi = -S_{\mu}^{ab} A_{\mu}^{ab}$$

where the curl brackets indicate the *anti-commutator*

$$\{ \gamma^c, \Sigma_{ab} \} = 2 \epsilon_{abcd} \gamma^5 \gamma^d$$

$$S_{\mu}^{ab} = -\frac{1}{4} \epsilon_{ab}^{cd} \epsilon_{\mu}^{\, c} \gamma^d$$

- $j^d_{(A)} = \bar{\psi} \gamma_5 \gamma^d \psi$: *spinor axial current* interacting with the gauge field $A_{\mu}$
Explicit form of the interaction Lagrangian density:

\[ \mathcal{L}_{int} = \frac{1}{4} \overline{\psi} \epsilon_{abcd}^{c} \gamma^{5} \gamma^{d} A_{c}^{ab} \psi \]

\[ a = \{0, \alpha\}: \text{split of the gauge field} \rightarrow A_{0}^{0\alpha}, A_{0}^{0\beta}, A_{0}^{0\gamma}, A_{0}^{0\gamma} \]

- We impose the \textit{time-gauge} associated to this picture: \( A_{0}^{0\beta} = 0 \)
- \( A_{0}^{0\alpha} \) is saturated on the completely anti-symmetric symbol \( \epsilon_{0\alpha\delta}^{0} \equiv 0 \)

Now we get

\[ \mathcal{L}_{int} = \frac{1}{4} \overline{\psi} \left( \epsilon_{0\alpha\delta}^{0} \gamma^{5} \gamma^{5} A_{0}^{0\alpha} + \epsilon_{\alpha\beta0}^{0} \gamma^{5} A_{0}^{0\gamma} \right) \psi \]
The total Lagrangian density rewrites as

\[ \mathcal{L} = \frac{i}{2} \left( \psi^\dagger \gamma^0 \gamma^a \partial_a \psi - \partial_a \psi^\dagger \gamma^0 \gamma^a \psi \right) - m \gamma^0 \psi^\dagger \psi + 
+ \psi^\dagger C_0 \gamma^0 \gamma_5 \gamma^0 \psi + \psi^\dagger C_\alpha \gamma^0 \gamma_5 \gamma^\alpha \psi \]

with the identifications:

\[ C_0 = \frac{1}{4} \epsilon^{\gamma}_{\alpha\beta\gamma} A_\gamma^{\alpha\beta} \quad C_\alpha = \frac{1}{4} \epsilon^{\gamma}_{0\beta\alpha} A_\gamma^{0\beta} \]

Finally, from \( \delta S = 0 \) variation wrt \( \psi^\dagger \) leads to the modified Dirac eq

\[ (i \gamma^0 \gamma^0 \partial_0 + C_\alpha \gamma^0 \gamma_5 \gamma^\alpha + i \gamma^0 \gamma^\alpha \partial_\alpha + C_0 \gamma^0 \gamma_5 \gamma^0 ) \psi = m \gamma^0 \psi \]
We look now for **stationary solution** of the Dirac equation: 
\[ \psi(x, t) \rightarrow \psi(x) \ e^{-i\mathcal{E}t} \]

Using the **Standard Representation** of Dirac matrices:

\[
\psi = \begin{pmatrix} \chi \\ \phi \end{pmatrix} \quad \psi^\dagger = (\chi^\dagger, \phi^\dagger)
\]

\[
\gamma^\alpha = \begin{pmatrix} 0 & \sigma_\alpha \\ -\sigma_\alpha & 0 \end{pmatrix} \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

the 2-component spinors \( \chi \) and \( \phi \) are found to satisfy the two coupled eqs

\[
(\mathcal{E} - \sigma_\alpha C^\alpha) \chi - (\sigma_\alpha p_\alpha + C_0) \phi = m \chi
\]

\[
(\mathcal{E} - \sigma_\alpha C^\alpha) \phi - (\sigma_\alpha p_\alpha + C_0) \chi = -m \phi
\]
In order to investigate the **non-relativistic limit** $\rightarrow \mathcal{E} = E + m$

The coupled equations rewrite now

\[
(E - \sigma_{\alpha} C^{\alpha}) \chi = (\sigma^{\alpha} p_{\alpha} + C_0) \phi
\]

\[
(E - \sigma_{\alpha} C^{\alpha} + m) \phi = (\sigma^{\alpha} p_{\alpha} + C_0) \chi - m \phi
\]

• In the non-relativistic limit both $|E|$ and $|\sigma_{\alpha} C^{\alpha}|$ terms **are small** in comparison wrt the mass term $m$:

\[
\phi = \frac{1}{2m} (\sigma^{\alpha} p_{\alpha} + C_0) \chi
\]

$\phi$ is smaller than $\chi$ by a factor of order $p/m$ (i.e. $v/c$): **small components**
Fermion dynamics in the non-relativistic limit: a generalized Pauli Equation

Using the Pauli matrix relation \((\sigma \cdot A)(\sigma \cdot B) = A \cdot B + i\sigma \cdot (A \times B)\) we can combine the two eqs in the following expression

\[
E \chi(x) = \frac{1}{2m} \left[ p^2 + C_0^2 + 2C_0 (\sigma^\alpha p_\alpha) + \sigma_\alpha C^{\alpha} \right] \chi(x)
\]

Strong analogies with the electro-magnetic case: Pauli Equation

\[
E \chi(x) = \frac{1}{2m} \left[ (p + A)^2 + \mu_B \sigma \cdot B + \Phi \right] \chi(x)
\]

where \(\mu_B = e/2m\) is the Bohr magneton and \(A\) is the vector potential.

→ It is worth noting the presence of a term related to the helicity of the 2-spinor: this coupling is controlled by the rotation-like component associated to \(C_0\).

→ A Zeeman-like coupling associated to the boost-like component \(C^{\alpha}\) is also present.
Let us now neglect the term $C_0^2$ and implement the symmetry

$$\partial_\mu \rightarrow \partial_\mu + A^{U(1)}_\mu + A_{\mu}^{ab} \Sigma_{ab} \quad \text{with} \quad A = 0$$

→ we introduce a Coulomb central potential $V(r)$: $E \rightarrow E - V(r)$

$$H_0 = \frac{p^2}{2m} - \frac{Ze^2}{(4\pi\bar{\epsilon}_0)r}$$

$$H' = \frac{1}{m} \left[ 2C_0 (S_\alpha p^\alpha) + S_\alpha C^\alpha \right]$$

where $S$ is the spin operator: $\sigma_\alpha = 2S_\alpha$.

*These Hamiltonians characterize the electron dynamics in a hydrogen-like atom in presence of a gauge field of the Lorentz Group.*
Generalization to curved space-time and the role of the torsion field

First Order Approach: in presence of torsion $T^\rho_{\mu\nu}$ (Riemann-Cartan space $U^4$), the II Cartan Structure eq writes

$$\partial_\mu e_\nu^a - \partial_\nu e_\mu^a - \tilde{\omega}_\mu^{ab} e_\nu^b + \tilde{\omega}_\nu^{ab} e_\mu^b = e^a_\rho T^\rho_{\mu\nu} = T^a_{\mu\nu}$$

- The connections, solutions of the Cartan eq, are

$$\tilde{\omega}_\mu^{ab} = \omega_\mu^{ab} + K_\mu^{ab}$$

Here $K_\mu^{ab}$ is the contortion field

$$K_\nu^\mu = -\frac{1}{2}(T^\mu_{\nu\rho} - T^\mu_{\rho\nu} + T^\mu_{\nu\rho})$$

and $\omega_\mu^{ab}$ are the usual Riemannian spin connections.

→ These connections do not describe any physical field: $K_\mu^{ab}$ appear only in a non-dynamical term: in presence of fermionic matter, substituting $\tilde{\omega}_\mu^{ab}$ in the HE action, $K_\mu^{ab}$ become proportional to the spin density (Einstein-Cartan model).
Total connections can be rewritten as: 

\[ C_{\mu}^{ab} = \tilde{\omega}_{\mu}^{ab} + A_{\mu}^{ab} \]

- \( A_{\mu}^{ab} \): Lorentz gauge connections connected with the appearance of torsion

Interaction term between the spin connections \( \omega \) and the fields \( A \):

(writes in a more compact formalism)

\[
S_{int} = 2\int \epsilon_{abcd} e^a \wedge e^b \wedge \omega^c \wedge A^d[
\]

\[
S (e, \omega, A, \psi, \bar{\psi}) = \frac{1}{4} \int \epsilon_{abcd} e^a \wedge e^b \wedge R^{cd} +
- \frac{1}{32} \int tr \, \star \, F \wedge F - \frac{1}{4} \int \epsilon_{abcd} e^a \wedge e^b \wedge \omega^c \wedge A^d[
+ \frac{1}{2} \int \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge \left[ i\bar{\psi} \gamma^d \left( d - \frac{i}{4} (\omega + A) \right) \psi - i \left( d + \frac{i}{4} (\omega + A) \right) \bar{\psi} \gamma^d \psi \right]
\]
Variations:

- No fermion matter:

\[
d^a \omega^a = A_a^b \wedge e^b
\]

\[
\tilde{\omega}_\mu^{ab} = \omega_\mu^{ab} + A_\mu^{ab}
\]

→ \( A_\mu^{ab} \) can be identified with the contortion field \( K_\mu^{ab} \) (II Cartan eq)

- With fermion matter:

\[
\tilde{\omega}_\mu^{ab} = \omega_\mu^{ab} + A_\mu^{ab} + \frac{1}{4} e^a_{bcd} e_\mu^c \tilde{j}^d(A)
\]

→ spin density results to be the source term of the Yang-Mills eq for the new Lorentz connections.
Conclusions

• A gauge theory of the Lorentz group is developed starting from the ambiguity in dealing with isometric diffeomorphism-induced Lorentz transformations. The not clear behaviors under local transformations of fermion field and spin connections allows to introduce new connections for the model.

• In order to restore the invariance under diffeo-induced local Lorentz \(trs\) of a spinor lagrangian in a flat scape-time, we need to introduce a new gauge field behaving like a Yang-Mill field. \textit{Spin connections are assumed to behave like coordinate vectors and are not gauge fields.}

• The analysis of the spinor interaction lagrangian in presence of the new gauge field, in the non-relativistic limit, leads to a Pauli-like equation describing the behavior of the large components of a 4-spinor.

• The generalization in curved space-time allows to identify the Lorentz gauge field with the tetradic projection of the \textit{contortion field} arising from the Second Cartan Structure equation.