Extended Approach to the Canonical Quantization in the Minisuperspace

**Motivations:**
Find a unique framework which phenomenologically describes the effective Friedmann dynamics of LQC and braneworlds scenario. Deformed minisuperspace (Heisenberg) algebra which is related to the $\kappa$-Pincaré one. This way, a non-trivial link between these theories is founded from a low energy perspective.

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Summary of talk

I. Deformed quantum mechanics

II. Deformed FRW dynamics

III. Deformed uncertainty principle

IV. Discussion and conclusions
I. Deformed quantum mechanics

Deformed Heisenberg algebra such that:
(i) No deforms rotation and translation groups
(ii) Ordinary Lie one is recovered in a limit

\[
[q_i, p_j] = i\delta_{ij}\sqrt{1 \pm \alpha p^2} \quad p^2 = p_i p^i \quad (1)
\]
\[
[p_i, p_j] = 0
\]
\[
[q_i, q_j] = \mp i\alpha J^{\alpha}_{ij}
\]

Generators of rotation group

\[
J^{\alpha}_{ij} = \frac{1}{\sqrt{1 \pm \alpha p^2}}(q_i p_j - q_j p_i) \quad (2)
\]

In 3-dim:
\[ J \in SU(2), \; [J^{\alpha}_{ij}, q_j] = i\epsilon_{ijk}q_k, \; [J^{\alpha}_{ij}, p_j] = i\epsilon_{ijk}p_k \]

No sign in (1) is selected at all by the assumptions

Algebra (1) can be obtained considering:
- q as a suitably $\kappa$-deformed Newton-Wigner position operator
- p, J as the generators of translations and rotations of $\kappa$-Poincaré algebra
1-dim:

\[ [q, p] = i\sqrt{1 \pm \alpha p^2} \]  \hspace{1cm} (3)

\( p \in \mathbb{R} \) in the (\(+\))-sector

\( p \in I(-1/\sqrt{\alpha}, 1/\sqrt{\alpha}) \) in the (\(-\))-sector

Representation algebra: momentum space

\[ p\psi(p) = p\psi(p) \]  \hspace{1cm} (4)

\[ q\psi(p) = i\sqrt{1 \pm \alpha p^2} \partial_p \psi(p) \]

Hilbert spaces \( p, q \) self-adjoint operators:

\[ \mathcal{F}_\pm = L^2 \left( \mathbb{R}(I), dp/\sqrt{1 \pm \alpha p^2} \right) \]  \hspace{1cm} (5)

Hilbert spaces unitarily inequivalent \( \Rightarrow \)

different physical predictions

For \( \alpha \to 0 \) the ordinary one \( L^2(\mathbb{R}, dp) \) is recovered in both the cases
Harmonic oscillator: \( \mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2q^2 \)

Considering the above representation (4)

\[
(\hat{\mathcal{H}} - E)\psi(p) = 0
\] (6)

Mathieu equation \( \Rightarrow \psi(p) \) in terms of Mathieu cosine and sine

Spectrum at the lowest order \((\sqrt{\alpha/d} \ll 1)\):

\[
E_n = \frac{\omega}{2}(2n + 1) \pm \frac{\omega}{8}(2n^2 + 2n + 1) \left( \frac{\alpha}{d^2} \right)
\] (7)

Characteristic length scale \( d = 1/\sqrt{m\omega} \)

\( E_n^{(-)} \) corresponds to the spectrum of the h.o. in polymer quantum mechanics

\( E_n^{(+)} \) spectrum of GUP \([q, p] = i(1 + \beta p^2)\)
II. Deformed FRW dynamics

The FRW cosmological models

\[ ds^2 = -N^2 dt^2 + a^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \]  (8)

\[ N = N(t) \] is the lapse function
\[ a = a(t) \] the scale factor

**Isotropy: phase space of GR 2-dim**
\[ \Gamma = (a, p_a) \]

Scalar constraint:

\[ \mathcal{H} = -\frac{2\pi G p_a^2}{3a} - \frac{3}{8\pi G} a k + a^3 \rho = 0 \]  (9)

\[ \rho = \rho(a) \] denotes a generic energy density

\[ \rho \sim 1/a^4 \] ultra-relativistic gas
\[ \rho \sim 1/a^5 \] perfect gas
\[ \rho \sim \text{const} \] cosmological constant
Extended Hamiltonian:

\[ \mathcal{H}_E = \frac{2\pi G}{3} N \frac{p_a^2}{a} + \frac{3}{8\pi G} Nak - Na^3 \rho + \lambda \pi \] (10)

\( \lambda \) is a Lagrange multiplier

\( \pi \) momenta conjugate to \( N \), i.e. it vanishes

Equations of motion:

\[ \dot{N} = \{N, \mathcal{H}_E\} = \lambda, \quad \dot{\pi} = \{\pi, \mathcal{H}_E\} = \mathcal{H} \] (11)

Primary constraint \( \pi = 0 \) satisfied at all times

\[ \Rightarrow \text{scalar (secondary) constraint } \dot{\pi} = \mathcal{H} = 0 \]

Dynamics of coordinate \( a \) and momenta \( p_a \) depends on the symplectic structure

If \( \{a, p_a\} = 1 \) we obtain Friedmann equation

\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} \] (12)
Modified symplectic geometry

Parameter $\alpha$ independent with respect $\hbar$

$$-i[q, p] \implies \{q, p\}_\alpha = \sqrt{1 \pm \alpha p^2} \quad (13)$$

Deformed Poisson bracket:

$$\{F, G\}_\alpha = \{F, G\}\{q, p\}_\alpha = \{F, G\}\sqrt{1 \pm \alpha p^2} \quad (14)$$

This is anti-symmetric, bilinear and satisfies the Leibniz rules as well as the Jacobi identity

Time evolution of $q, p$ with respect to $\mathcal{H}$

$$\dot{q} = \{q, \mathcal{H}\}_\alpha = \frac{\partial \mathcal{H}}{\partial p} \sqrt{1 \pm \alpha p^2} \quad (15)$$

$$\dot{p} = \{p, \mathcal{H}\}_\alpha = -\frac{\partial \mathcal{H}}{\partial q} \sqrt{1 \pm \alpha p^2}$$
FRW deformed phase space

Fundamental commutator:

\[ \{a, p_a\}_\alpha = \sqrt{1 \pm \alpha p_a^2} \quad (16) \]

Equation of motion with respect to \( \mathcal{H}_E \)

\[ \dot{a} = \{a, \mathcal{H}_E\}_\alpha = \frac{4\pi G}{3} N p_a \frac{p_a}{a} \sqrt{1 \pm \alpha p_a^2} \quad (17) \]

\[ \dot{p}_a = \{p_a, \mathcal{H}_E\}_\alpha = N \left( \frac{2\pi G p_a^2}{3 a^2} - \frac{3}{8\pi G} k + 3a^2 \rho + a^3 \frac{d\rho}{da} \right) \sqrt{1 \pm \alpha p_a^2} \quad (18) \]

Deformed Friedmann equation

\[ H^2 = \left( \frac{8\pi G}{3} \rho - \frac{k}{a^2} \right) \left[ 1 \pm \frac{3\alpha}{2\pi G} a^2 \left( a^2 \rho - \frac{3}{8\pi G} k \right) \right] \quad (19) \]
FRW flat \((k = 0)\) model

\[
H_{k=0}^2 = \frac{8\pi G}{3} \rho \left( 1 \pm \frac{\rho}{\rho_{\text{crit}}} \right)
\]  

(20)

critical density \(\rho_{\text{crit}} = \frac{2\pi G}{3\alpha} \rho_P\).

We have assumed the existence of a fundamental scale, i.e. \(\rho \leq \rho_{\text{crit}}\)

For \(\alpha \to 0\), \(\rho_{\text{crit}} \to \infty \Rightarrow \) ordinary behavior

\((-)\)-equation equivalent to LQC

\((+)\)-equation equivalent to braneworlds

The \((-)\) sign implies a bouncing cosmology, while with the \((+)\) one \(\dot{a}\) can not vanish

A \((-)\)-braneworlds scenario appears if the extra-dimension is time-like (open question)
III. Deformed uncertainty principle

Uncertainty principle related to (16)

$$\Delta a = \frac{1}{2} \left| \left( \frac{1 \pm \alpha \langle p_a \rangle^2}{(\Delta p_a)^2} \pm \alpha \right)^{1/2} \right|$$

(21)

For $\Delta p_a \gg (\Delta p_a)^* \equiv \sqrt{(1 \pm \alpha \langle p_a \rangle)/\alpha}$

minimal uncertainty in the scale factor $\Delta a_0 = \sqrt{\alpha}/2$

**Brane-framework, (+)-sector:** $\Delta a_0 \neq 0$ is a global minimum (No physical states which are position eigenstates exist at all)

**LQC-framework, (−)-sector:** $\Delta a_0 = 0$ appears for $\Delta p_a = (\Delta p_a)^* \propto 1/\sqrt{\alpha}$, i.e. when the deformation energy is reached
IV. Discussion and conclusions

- A unique framework which phenomenologically describes both the effective Friedmann evolution of LQC and branewords models is obtained by the use of a deformed Heisenberg algebra.

- The algebra leaves undeformed the translation group and preserves the rotational invariance. Furthermore is related to the $\kappa$-Poincaré one and no sign in the deformation term is selected at all.

- The brane-deformed scenario is such that a minimal uncertainty in the scale factor appears. On the other hand, in the loop one, we have the vanishing uncertainty when the deformed energy is reached.